Vector Bundles

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Contents

1 Introduction and Basic Notions

1.1 Motivation

Perhaps the simplest way to construct a new manifold out of ones you are familiar with is to simply take two of your favorite manifolds and then consider their Cartesian product. Let's you start with two smooth manifolds X and Y of dimension n and m respectively, then it's not difficult to give $X \times Y$ the structure of smooth $n+m$ dimensional manifold. Furthermore, understanding the differential topology of this new product manifold often comes down to just understanding each factor individually. For example, tangent spaces and differentials split over product manifolds in exactly the way you would want, namely

$$T_{(x,y)}(X \times Y) \simeq T_xX \times T_yY$$

and if we have smooth maps $f : X \to A$ and $g : Y \to B$, then

$$d(f \times g)_{(x,y)} = df_x \times dg_y.$$

To define manifolds, we fix some space we have a good understanding of (Euclidean space), and consider other spaces which may be globally quite complicated, but at least locally look like the space we are familiar with. Embracing this philosophy, we might take the next step and fix *two spaces* B and F, and then consider the class of spaces which "locally look like $B \times F$." Making this idea precise leads us to the topological notion of a fiber bundle.

1.2 The Basic Setup

Definition 1.1. A *fiber bundle* with base space B, total space E, and model fiber F, is a triple of topological spaces (E, B, F) together with a continuous surjection $\pi : E \to B$ such that

1. The fibers $\pi^{-1}(x)$ are homeomorphic to F for all $x \in X$.
2. For all $x \in X$, there exists a *trivializing neighborhood* U containing x and a homeomorphism $h : \pi^{-1}(U) \to U \times F$ such that the diagram

$$\begin{array}{ccc} \pi^{-1}(U) & \xrightarrow{h} & U \times F \\ {\scriptstyle \pi}\downarrow & \swarrow {\scriptstyle p_U} & \\ U & & \end{array}$$

commutes, where $p_U : U \times F \to U$ is projection onto the first coordinate.

We say that the fiber bundle E is *trivial* if it is just homeomorphic to the product $B \times F$. If instead of spaces and continuous maps we work with manifolds and smooth maps, we recover the notion of a *smooth fiber bundle*. From here on out we will primarily be working in the smooth setting. In the study of vector bundles, we only consider smooth bundles whose fibers are vector spaces. Although vector spaces are the simplest possible type of manifold, the theory of vector bundles is quite vast and has wide ranging applications across mathematics.

Definition 1.2. A *smooth vector bundle* is a smooth fiber bundle $\pi : E \to B$ such that all of the fibers $F = \pi^{-1}(b)$ have the structure of an $\mathbb{R}$-vector space in a way that is compatible with the local trivialization condition. In particular, we require that every point $b \in B$ has a locally trivializing neighborhood $U \subset B$ such that there is a diffeomorphism

$$g : U \times \mathbb{R}^{\dim \pi^{-1}(b)} \to \pi^{-1}(U)$$

so that g^{-1} satisfies part 2 of Definition 1.1 and for which the map $g(x, \cdot) : \mathbb{R}^{\dim \pi^{-1}(x)} \to \pi^{-1}(x)$ is a linear isomorphism for all $x \in U$.

Note that in this definition we don't require all the fibers to be diffeomorphic (be the same dimension). However, the dimension of the fiber is a continuous integer valued function on B, so it will be constant on connected components. We will primarily be focused on the case of connected base spaces so we will usually just assume all the fibers are vector spaces of the same dimension. A vector bundle where all the fibers have the same dimension k is referred to as a *rank k vector bundle*. We also often refer to local trivializations as *local coordinates* on the vector bundle.

Definition 1.3. Let $\pi : E \to B$ and $\pi' : E' \to B'$ be two smooth vector bundles. A *vector bundle (homo)morphism* from (E, B) to (E', B') is a pair of smooth maps $f : E \to E'$ and $g : B \to B'$ such that the diagram

$$\begin{array}{ccc} E & \xrightarrow{f} & E' \\ {\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle \pi'} \\ B & \xrightarrow{g} & B' \end{array}$$

commutes and for all $b \in B$ the map $f|_{\pi^{-1}(b)} : \pi^{-1}(b) \to {\pi'}^{-1}(g(b))$ is linear. We say that a vector bundle morphism is a *vector bundle isomorphism* if it has an inverse which is also a vector bundle morphism and we say that two vector bundles are isomorphic if there exists a vector bundle isomorphism between them.

Remark 1.4. If we fix a base manifold M, the collection of vector bundles over M equipped with vector bundle morphisms forms a category. We will say more about this later when we state the smooth version of the Serre-Swan Theorem.

Definition 1.5. Let $\pi : E \to B$ be a smooth vector bundle. A *(cross-)section* of (E, B) is a smooth map $s : B \to E$ such that $s(b) \in \pi^{-1}(b)$ for all $b \in B$. We say a section is *nowhere zero* or *non-vanishing* if $s(b)$ is not the zero vector in $\pi^{-1}(b)$ for all $b \in B$. As we will see later, whether a bundle admits a non-vanishing section reveals a lot about its topology.

Definition 1.6. (Pullback Bundles) Let $\pi : E \to B$ be a vector bundle and let $f : \tilde{B} \to B$ be a smooth map from some other manifold $\tilde{B}$. We can pull the bundle E over B back along f to a bundle $\tilde{E}$ over $\tilde{B}$ as follows: Let $\tilde{E}$ be the subset of $\tilde{B} \times E$ consisting of pairs $(\tilde{b}, e)$ where $f(\tilde{b}) = \pi(e)$. Define $\tilde{\pi} : \tilde{E} \to \tilde{B}$ by $\tilde{\pi}(\tilde{b}, e) = \tilde{b}$. Let $\hat{f} : \tilde{E} \to E$ be the map $\hat{f}(\tilde{b}, e) = e$. We can equip the fibers $\tilde{\pi}^{-1}(\tilde{b})$ with the structure of a vector space so that the map $\hat{f}$ restricted to the fiber is a vector space isomorphism to the fiber $\pi^{-1}(f(\tilde{b}))$.

To show that $\tilde{f} : \tilde{E} \to \tilde{B}$ is a vector bundle we need local trivializations. Suppose that U is a trivial neighborhood in B with trivializing diffeomorphism $h : U \times \mathbb{R}^n \to \pi^{-1}(U)$. Let $\tilde{U} = f^{-1}(U) \subset \tilde{B}$ and let $\tilde{h} : \tilde{U} \times \mathbb{R}^n \to \tilde{\pi}^{-1}(\tilde{U})$ be given by

$$\tilde{h}(\tilde{b}, x) = (\tilde{b}, h(f(\tilde{b}), x)).$$

The collection of these $\tilde{h}$'s provide local coordinates to show that $\tilde{\pi} : \tilde{E} \to \tilde{B}$ is indeed a smooth vector bundle. We will denote this bundle as $f^*(E)$ and call it the *pullback bundle*.

Proposition 1.7. *Let $\pi : E \to B$ be a vector bundle and let $f : X \to B$ be a smooth map. Then, $f^*(E)$ is a bundle on X such that f is part of a bundle homomorphism that maps fibers isomorphically onto fibers. Furthermore, any other bundle on X with this property must be isomorphic to $f^*(E)$.*

Proof. See [1] □

Definition 1.8. (Restricting Bundles) Let $\pi : E \to B$ be a vector bundle and let $C \subset B$ be a submanifold. We can restrict the bundle π to the sub-manifold C by taking the pullback bundle along the inclusion map $\iota : C \hookrightarrow B$.

Definition 1.9. (Sub-Bundles) Let $\pi : E \to B$ and $\pi' : E' \to B$ be vector bundles such that $E' \subset E$ and $(\pi')^{-1}(b)$ is a vector subspace of $\pi^{-1}(b)$ for all $b \in B$. Then, we say that E' is a sub-bundle of E.

Definition 1.10. (Product Bundles) If $\pi_1 : E_1 \to B_1$ and $\pi_2 : E_2 \to B_2$ are vector bundles then we can form a product bundle $\pi_1 \times \pi_2 : E_1 \times E_2 \to B_1 \times B_2$ in the obvious way.

Definition 1.11. (Quotient Bundles) Suppose E' is a sub-bundle of E over a space B. Then, we can form the quotient bundle E/E' where the fibers are the quotient vector spaces $\pi^{-1}(b)/(\pi')^{-1}(b)$ for all $b \in B$.

1.3 Transition Maps

In order to establish the most useful examples and constructions involving vector bundles, it is necessary to establish some theory about how to easily construct bundles without having to appeal to the original definition every time.

Definition 1.12. Let $\pi : E \to B$ be a vector bundle and let $U, V \subset B$ be two trivial neighborhoods with non-empty intersection and trivializing diffeomorphisms $\phi_U : \pi^{-1}(U) \to U \times \mathbb{R}^k$ and $\phi_V : \pi^{-1}(V) \to V \times \mathbb{R}^k$. We then get a diffeomorphism

$$\phi_U \circ \phi_V^{-1} : (U \cap V) \times \mathbb{R}^k \to (U \cap V) \times \mathbb{R}^k.$$

For each $x \in U \cap V$, the map $\phi_U \circ \phi_V^{-1}$ induces a linear isomorphism $g_{UV}(x) : \{x\} \times \mathbb{R}^k \to \{x\} \times \mathbb{R}^k$. This gives a smooth map $g_{UV} : U \cap V \to GL_k(\mathbb{R})$ which is called the *transition function*.

Remark 1.13. The group $GL_k(\mathbb{R})$ in the previous definition is an example of the more general notion of a *structure group* of a fiber bundle.

Proposition 1.14. *Let g_{UV} be as in the previous definition. Let W be a third open subset of B such that $U \cap V \cap W$ is non-empty. Then,*

1. $g_{UV}(x) \cdot g_{VU}(x) = Id_{\mathbb{R}^k}$ *for all* $x \in U \cap V$.
2. $g_{UV}(x) \cdot g_{VW}(x) \cdot g_{WU}(x) = g_{UW}(x)$ *for all* $x \in U \cap V \cap W$.

Theorem 1.15. *Let B be a manifold with an open cover $\{U_\alpha\}_{\alpha\in A}$ and a collection of smooth functions $g_{\alpha\beta} : U_\alpha \cap U_\beta \to GL_k(\mathbb{R})$ for every $\alpha, \beta \in A$ such that the collection of functions $\{g_{\alpha\beta}\}$ satisfies the identities of Proposition 1.14. Then, there exists a unique rank k vector bundle $\pi : E \to B$ with $\{U_\alpha\}$ a collection of trivial neighborhoods with transition functions $\{g_{\alpha\beta}\}$.*

Proof. We present a brief proof that skips some details. As a topological space, define E to be the set

$$\bigsqcup_{\alpha\in A} U_\alpha \times \mathbb{R}^k$$

modulo the relation that

$$(x, v) \in U_\alpha \times \mathbb{R}^k \sim (x, g_{\alpha\beta}(v)) \in U_\beta \times \mathbb{R}^k.$$

Because B already carries the structure of a smooth manifold, the inclusion maps

$$U_\alpha \times \mathbb{R}^k \hookrightarrow E$$

can be used to provide local parametrizations of E. The transition maps for these local parametrizations will be smooth because the transition maps for B are already known to be smooth. This gives E the structure of a smooth manifold. We also have a well defined projection map $\pi : E \to B$ where $\pi([x, v]) = x$. The fact that $\{g_{\alpha\beta}\}$ satisfies the identities of Proposition 1.14 guarantees that $\pi : E \to B$ is indeed a smooth vector bundle.

To show uniqueness, suppose $\widetilde{\pi} : \widetilde{E} \to B$ is a smooth vector bundle where $\{U_\alpha\}$ is a collection of trivial neighborhoods with transition functions $\{g_{\alpha\beta}\}$. For each U_α, there is a trivializing diffeomorphism $\tilde{h}_\alpha : \tilde{\pi}^{-1}(U_\alpha) \to U_\alpha \times \mathbb{R}^k$. We can define a set map $f : \widetilde{E} \to E$ by setting $f(x) = [\tilde{h}_\alpha(x)]$ where $x \in U_\alpha$ and $[\tilde{h}_\alpha(x)]$ is the equivalence class of $\tilde{h}_\alpha(x)$ in E. This map is well defined, independent of the choice of U_α, and bijective because E and $\tilde{E}$ have the same transition maps. The smooth structure on E is defined so that the inclusions $U_\alpha \times \mathbb{R}^k \to E$ are smooth. On each $\tilde{\pi}^{-1}(U_\alpha)$, f is given by a diffeomorphism followed by a smooth map and is thus smooth on these neighborhoods. Because $\tilde{\pi}$ is a surjection and $\{U_\alpha\}$ is an open cover of B, this shows that f is actually smooth on all of $\tilde{E}$. The collection of diffeomorphisms $\tilde{h}_\alpha^{-1} : U_\alpha \times \mathbb{R}^k \to \tilde{\pi}^{-1}(U_\alpha)$ defines a well defined smooth map out of $\bigsqcup_\alpha U_\alpha \times \mathbb{R}^k$ into $\tilde{E}$. Because the transition functions are the same for $\tilde{E}$ and E, this smooth map descends to a well-defined map $E \to \tilde{E}$ which is exactly f^{-1}. Since E is a quotient manifold of $\bigsqcup_\alpha U_\alpha \times \mathbb{R}^k$, f^{-1} is thus smooth and therefore f is a diffeomorphism. It also follows straightforwardly from our definition of f that f and f^{-1} are bundle morphisms. Therefore, E is isomorphic to $\tilde{E}$ as desired. ∎

Often times we will construct vector bundles by specifying an open cover $\{U_\alpha\}$ of B on which we declare the bundle to be trivial, and then defining transition functions which satisfy the identities of Proposition 1.14. This theorem tells us that this construction is a well defined method for defining vector bundles. We will use this method extensively in the next section to construct a plethora of useful examples of vector bundles.

1.4 Some Key Examples

As we will see later, any vector bundle over a contractible base space is trivial. Therefore, any coordinate neighborhood on the base space of a vector bundle will provide a trivializing neighborhood for that bundle. Therefore, to specify a bundle we usually define transition functions for an atlas of the base space. We will not verify it explicitly but straightforward computations show that all the transition maps we discuss satisfy the identities of Proposition 1.14.

Example 1. (Tangent Bundle) Let M be a smooth manifold of dimension n. We construct a rank n bundle TM over M where the fiber at each point $p \in M$ is the vector space T_pM. Let $\{(U_\alpha, \phi_\alpha)\}$ be a cover of M by open neighborhoods U_α with local parametrization diffeomorphisms $\phi_\alpha : \mathbb{R}^n \to U_\alpha$. We construct TM as a bundle using Theorem 1.15 by setting the transition maps $g_{\alpha\beta} : U_\alpha \cap U_\beta \to GL_n(\mathbb{R})$ to be

$$g_{\alpha\beta}(x) = d(\phi_\beta^{-1} \circ \phi_\alpha)_{\phi_\alpha^{-1}(x)}.$$

Example 2. (Dual Bundle) Let $\pi : E \to B$ be a vector bundle. We construct a new bundle $\pi^* : E^* \to B$ where each fiber $\pi^{*-1}(b)$ is the dual vector space of the fiber $\pi^{-1}(b)$. If $\{g_{\alpha\beta}\}$ are the transition functions for E, we define the transition functions $\{g^*_{\alpha\beta}\}$ for E^* by setting

$$g^*_{\alpha\beta}(x) = (g_{\alpha\beta}(x)^t)^{-1}.$$

Example 3. (Direct Sum, Tensor Product, Exterior Product, Symmetric Product) Because the direct sum, tensor product, symmetric product, and exterior product are operations on vector spaces which induce operations on linear maps, we can perform these operations to existing transition maps to get new ones defining new bundles. This procedure will be made more clear in the later section on continuous functors.

Example 4. (Normal Bundle) Let $M \subset \mathbb{R}^k$ be an embedded submanifold. We can define a bundle NM over M whose fibers are the orthogonal complement of T_pM inside $\mathbb{R}^k$. One way to construct this bundle is to consider it as the quotient bundle $T\mathbb{R}^k|_M/TM$.

1.5 Some Fundamental Theorems

The following is a short exposition of a very powerful theorem which will allow us to construct lots and lots of vector bundles from familiar vector space operations such as the direct product, tensor product, exterior product, and symmetric product.

1.5.1 Continuous Functor Constructions

Definition 1.16. Let $\mathcal{V}$ be the category of finite dimensional $\mathbb{R}$-vector spaces with morphisms as linear isomorphisms. A *continuous (smooth) functor in k-variables* is a covariant (or contravariant) functor $T : \mathcal{V} \times \cdots \times \mathcal{V} \to \mathcal{V}$ such that $T(f_1, \ldots, f_k)$ depends continuously (smoothly) on $f_1, \ldots, f_k$. This is well defined because $\hom(V, W)$ has a canonical choice of smooth structure for all $V, W \in \mathcal{V}$.

Example 5. Some key examples of smooth functors are:

1. The direct sum $V \oplus W$.
2. The tensor product $V \otimes W$.
3. The exterior product $\bigwedge^k(V)$.
4. The symmetric product $Sym^k(V)$.
5. The dual space V^*.
6. The hom functor which takes V, W to the vector space $\hom(V, W)$.

Any operation on vector spaces which is a continuous functor can be performed on vector bundles to get new unique well defined vector bundles. This is summed up in the following theorem.

Theorem 1.17. *Let $\pi_i : E_i \to B$ be a collection of k vector bundles with fibers $F|^i_b := \pi_i^{-1}(b)$. Let $T : \mathcal{V} \times \cdots \times \mathcal{V} \to \mathcal{V}$ be a smooth functor in k variables. Define $F|_b = T(F|^1_b, \ldots, F|^k_b)$ and let*

$$E = \bigsqcup_{b \in B} F|_b.$$

Then, there exists a canonical topology and smooth structure on E which makes it into a vector bundle over B with fibers $F|_b$.

Proof. See [3] □

1.5.2 Serre-Swan

Definition 1.18. Let $\pi : E \to B$ be a vector bundle. Define $\Gamma(E)$ to be the set of sections of $s : B \to E$.

The set $\Gamma(E)$ can naturally be given the structure of an $\mathbb{R}$-vector space since the fibers are vector spaces. However, we can not only scale sections by numbers but we can also scale them by smooth functions. For $f \in \mathcal{C}^\infty(B)$ and $s \in \Gamma(E)$, we can define a new section $f \cdot s \in \Gamma(E)$ by setting

$$f \cdot s(b) = f(b)s(b)$$

for all $b \in B$. This operation in fact gives $\Gamma(E)$ the structure of a $\mathcal{C}^\infty(B)$-module.

Definition 1.19. Fix a base manifold M and let E, E' be two bundles over M with a bundle morphism $f : E \to E'$. We get an induced map $f_* : \Gamma(E) \to \Gamma(E')$ by setting

$$f_*(s) = f \circ s.$$

It turns out that this map is in fact a homomorphism of $\mathcal{C}^\infty(M)$-modules.

Although the module $\Gamma(E)$ will not be free in general, it will always be nice in that it will always be finitely generated and projective.

Proposition 1.20. *The operation Γ which takes vector bundles to their $\mathcal{C}^\infty(M)$-module of sections is a functor from the category of vector bundles over M to the category of $\mathcal{C}^\infty(M)$-modules. Furthermore, $\Gamma(E)$ is always a finitely generated projective $\mathcal{C}^\infty(M)$ module.*

The theory of vector bundles appears naturally in three main contexts: algebraic geometry, topology, and the theory of smooth manifolds. Each of these contexts have their own version of a "Serre-Swan" style theorem which related vector bundles to modules over a ring of functions. We will state all three here to show their similarities but we have really only established the terminology to state the smooth version.

Theorem 1.21. *(Algebraic Version by Serre) If R is the coordinate ring of an affine algebraic variety X over a field, then the category of finitely generated projective R modules is equivalent to the category of algebraic vector bundles on X.*

Theorem 1.22. *(Topology Version by Swan) Let X be a compact Hausdorff space and let $C(X)$ be the ring of continuous functions $f : X \to k$ where k is either $\mathbb{C}$, $\mathbb{R}$, or the quaternions. Then, the category of finite rank k-vector bundles on X is equivalent to the category of finitely generated projective $C(X)$-modules via the section functor.*

Theorem 1.23. *(Smooth Version) Let M be a manifold. Then, the section functor Γ from the category of finite rank vector bundles over M to the category of finitely generated projective $\mathcal{C}^\infty(M)$-modules is an equivalence of categories.*

One interesting consequence of these theorems is a nice connection between the fact that any vector bundle is a direct summand of a trivial bundle and the fact that every projective module is a direct summand of a free module.

1.5.3 Homotopy Properties

In this section we will present some useful homotopy theoretic properties of vector bundles that will be useful in the next section. Throughout the following we let I denote the standard unit interval $I = [0, 1]$.

Proposition 1.24. *Let $\pi : E \to B \times [a, b]$ be a vector bundle. Then, E is trivial if and only if there is a $c \in (a, b)$ such that E restricted to $X \times [a, c]$ and E restricted to $X \times [c, d]$ are both trivial.*

Proposition 1.25. *For a bundle $\pi : E \to B \times I$ there exists an open cover $\{U_\alpha\}$ of B such that the restrictions of E to $U_\alpha \times I$ are all trivial.*

Theorem 1.26. *Let $\pi : E \to B \times I$ be a vector bundle. Then, E restricted to $B \times \{0\}$ is isomorphic to E restricted to $B \times \{1\}$.*

Theorem 1.27. *Homotopic maps induced isomorphic pullback bundles.*

Corollary 1.28. *Every vector bundle over a contractible space is trivial.*

2 Classifying Vector Bundles

2.1 The Clutching Construction

Because S^n can be covered by two charts whose intersection is homotopy equivalent to S^{n-1} (open overlapping upper and lower hemispheres), specifying a rank k vector bundle on S^n amounts to specifying a transition function $f : S^{n-1} \to GL_k(\mathbb{R})$. Such an f is called a *clutching function*. We denote the vector bundle on S^n induced by a clutching function f as E_f.

Theorem 2.1. *Let $f, g : S^{n-1} \to GL_k(\mathbb{R})$ be homotopic. Then E_f is isomorphic to E_g.*

Proof. Let $H : S^{n-1} \times [0, 1] \to GL_k(\mathbb{R})$ be a homotopy from f to g. Then, H will serve as a transition map to determine a vector bundle E_H on $S^{n-1} \times [0, 1]$. This vector bundle restricted to $S^{n-1} \times \{t\} \simeq S^{n-1}$ will have clutching function $H(\cdot, t)$. In particular, E_H restricted to $S^{n-1} \times 0$ will be E_f and E_H restricted to $S^{n-1} \times 1$ will be H_g. Therefore, by Theorem 1.26, E_f is isomorphic to E_g. □

Theorem 2.2. *A homotopy class of clutching functions $[f] : S^{n-1} \to GL_k(\mathbb{C})$ determines a unique isomorphism class of rank k complex vector bundles on S^n.*

Definition 2.3. An orientation on a vector bundle is a choice of orientation on all of the fibers so that the local trivialization maps are orientation preserving with respect to the standard orientation on $\mathbb{R}^n$.

The clutching functions for oriented bundles on S^n will be given by maps $f : S^{n-1} \to GL_k(\mathbb{R})^+$.

Theorem 2.4. *A homotopy class of maps $[f] : S^{n-1} \to GL_k(\mathbb{R})^+$ determines a unique isomorphism class of oriented real vector bundles on S^n.*

Corollary 2.5. *Every complex vector bundle on S^1 is trivial and for every rank k there are exactly two isomorphism classes of rank k vector bundles on S^1.*

Proof. Homotopy classes of maps from a single point to a space exactly specify path connected components of that space. The corollary follows from the previous two theorems and the fact that $GL_n(\mathbb{C})$ is path connected while $GL_n(\mathbb{R})$ has two path components. □

2.2 Classifying Vector Bundles on S^2

The previous theorem tells us that classifying vector bundles on S^2 amounts to classifying homotopy classes of maps $f : S^1 \to GL_k(\mathbb{R})$. Luckily for us, the fundamental group gives us a nice classification of all such maps. The theory of covering spaces and the fundamental group tells us that $\pi_1(GL_k(\mathbb{R})^+) = \mathbb{Z}/2\mathbb{Z}$ for all $k > 2$ and $\pi_1(Gl_2(\mathbb{R})^+) = \pi_1(S^1) = \mathbb{Z}$. Therefore, every real line bundle on S^2 is trivial, there are countably many isomorphism classes of rank 2 orientable bundles on S^2, and there are exactly 2 isomorphism classes of oriented rank k bundles on S^2 for $k > 2$.

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