Plateau's Problem and the Geometry of Multivarifolds

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1 Introduction To Plateau's Problem: Why Varifolds?

The classical Plateau problem asks the following question: Given a closed curve γ in three dimensional space, does there exist among all of the two dimensional surfaces with boundary γ, one with the least area? This question was originally motivated by the physical study of soap bubbles since various physical principles involving surface tension imply that soap films will form in such a way as to have the least possible area. During his nineteenth century experiments, the Belgian physicist Joseph Plateau observed that every shape of wire he could make, when dipped into soap, would bound such a surface with minimal area. This naturally led mathematicians to ask whether the purely mathematical formulation of the problem as previously stated has a solution. Some versions of the Plateau problem have been fully resolved, whereas some variations, especially higher dimensional ones, still remain open. The difficulty of rigorously formulating and solving the Plateau problem inspired many key developments in the Calculus of Variations and Geometric Measure Theory, some of which we will explore in this paper.

In particular, we will present a solution to a generalized higher dimensional analogue of Plateau's problem which uses the notion of a *multivarifold*. A multivarifold is an intrinsically stratified object made up of pieces of different dimensions, each of which acts like a generalized surface of that dimension. During some minimization procedures in higher dimensions, surfaces might have pieces which pinch off and become degenerate by going down in dimension. They could also develop singularities and self-intersections which prevent them staying as true manifolds. Multivarifolds are well suited to the Calculus of Variations and minimization problems because they are able to keep track of degenerate pieces and do not have the regularity restrictions of a manifold.

In this paper we will begin by introducing the theory of multivarifolds. We will then present and solve a multivarifold formulation of the Plateau problem which was first stated and solved by Dao Trong Thi in [3] and [4].

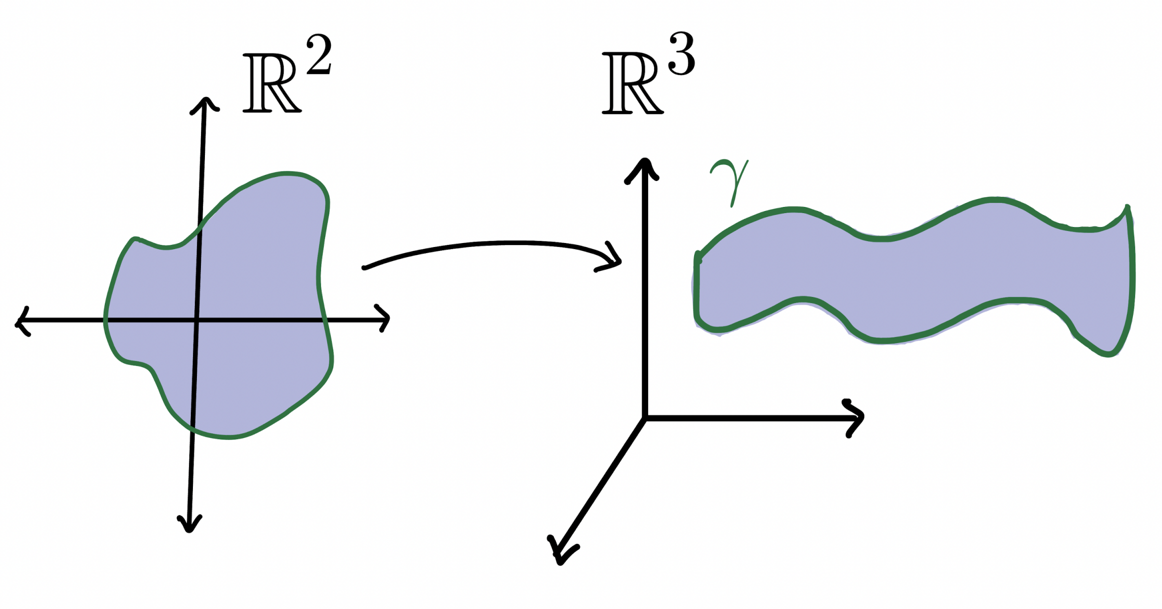

Figure 1: In the classical Plateau problem one is given a fixed contour γ and asked to find the parametrized surface with boundary γ that has the minimum possible surface area. This diagram shows a parameterization using a disk, however the classical Plateau problem does *not* put any requirement on the topological type of the surface so parameterized surfaces of any genus are allowed.

2 The Theory Multivarifolds

Going forward we will occasionally adopt the Bourbaki approach to theory of Radon measures on locally compact spaces as presented in [1]. This approach defines a Radon measure on a locally compact space X to be a linear functional on $C_0(X)$ (the space of continuous real valued functions on X with compact support) which is continuous with respect to the topology of uniform convergence on compact sets. We will most often adopt this approach in order to define measures μ by specifying what $\int f d\mu$ is for all $f \in C_0(X)$. We will therefore also freely interchange the notation $\int f d\mu$ with the notation $\mu(f)$.

2.1 Constructing Multivarifolds

2.1.1 What is a Multivarifold *really*?

Our goal is to construct a generalized notion of 'surface' or 'submanifold' which is as flexible as possible while maintaining the useful notions of tangent space, dimension, and volume. To do this we will take a measure theoretic approach which sacrifices the nice topological and regularity properties of smooth manifolds in favor of generality and useful variational properties. To construct multivarifolds, we will take the space of pairs (p, W) where p is a point in Euclidean space (or a Riemannian manifold) and W is a vector subspace of the tangent space at p. We will then define multivarifolds to be Radon measures on this space.

Because the Plateau problem is an inherently geometric question, its important to understand how this measure theoretic definition relates to geometry. Since we are defining multivarifolds to be a measure, we can look at their support, i.e. 'the points where the measure lives'. The support of a multivarifold will be a set of pairs (p, W) of points and tangent subspaces. We can think of the 'geometric content' of the multivarifold as being defined by the subset of Euclidean space (or manifold) of points which are in pairs that are part of the support of the measure. If p is such a point and (p, W) is such a pair, where $\dim W = d$, then we can think of p as being a 'd-dimensional point' with 'tangent space' W. For each dimension d we can collect the

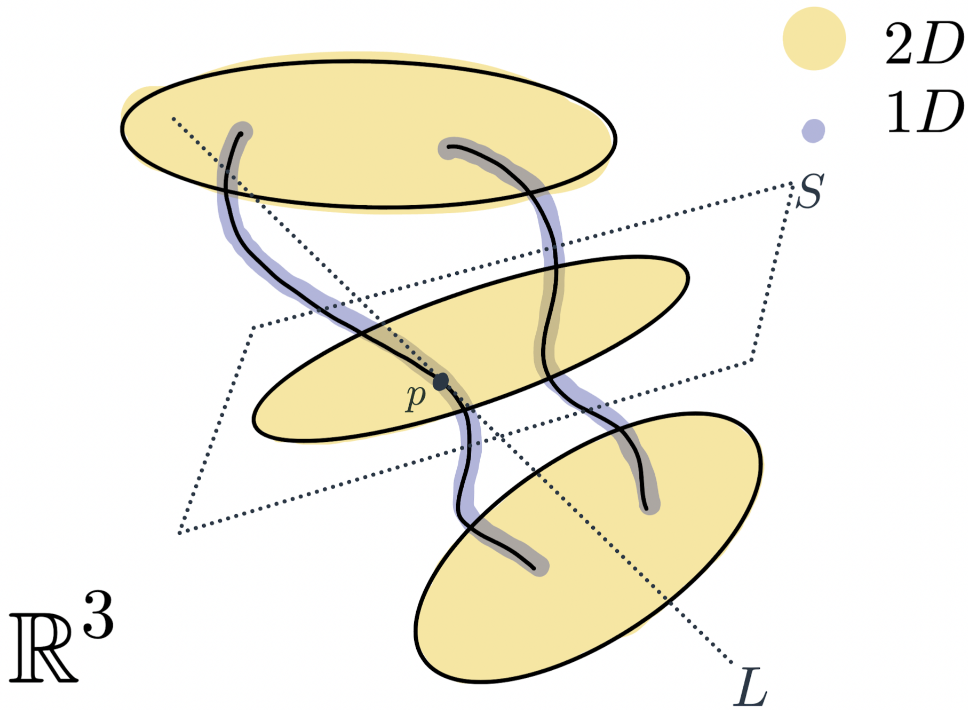

Figure 2: Example of the support of a Multivarifold in $\mathbb{R}^3$. The yellow disks are the two dimensional strata and the blue curves are the one dimensional strata. The point p illustrates how points can have tangent spaces of different dimensions included in the same multivarifold. Both (p, L) and (p, S) are in the support of the measure corresponding to this multivarifold.

set of 'd-dimensional points' into what is called the d-dimensional stratum of the multivarifold. If (p, W) and (p, W') are in the support of our measure and $\dim W' = d' \neq \dim W$, then p is a part of two different strata. For example, in Figure 2 the point p is part of the one-dimensional stratum corresponding to the blue curve as well as the two-dimensional one corresponding to the yellow disk.

In order to formalize these notions it's important to first recall the concept of a fiber bundle for topological spaces.

Definition 2.1. (Fiber Bundles) Let M be a topological space. The data of a *fiber bundle over M with model fiber F* is a topological space T, called the total space, and a projection map $\pi : T \to M$ which is a continuous surjection such that for all $x \in M$ there exists a neighborhood U_x of x such that $\pi^{-1}(U_x)$ is homeomorphic to $M \times F$. Intuitively, a fiber bundle over M with model fiber F is a topological space T which locally looks like $M \times F$.

The most common examples of fiber bundles in differential geometry are vector bundles over a manifold M, i.e. for all $p \in M$ the fiber $\pi^{-1}(p)$ is endowed with a vector space structure. The most prominent example being the tangent bundle TM. In order to construct our notion of a 'generalized lower dimensional submanifold' we need a more sophisticated bundle than the tangent bundle. We first introduce the crucial notion of a Grassmanian.

Definition 2.2. (Grassmanian) For any natural numbers $0 \leq k \leq n$ we define $\Gamma_k(n)$ to be the set of k-dimensional sub-spaces of $\mathbb{R}^n$. The set $\Gamma_k(n)$ can be endowed with the topology of a smooth manifold of

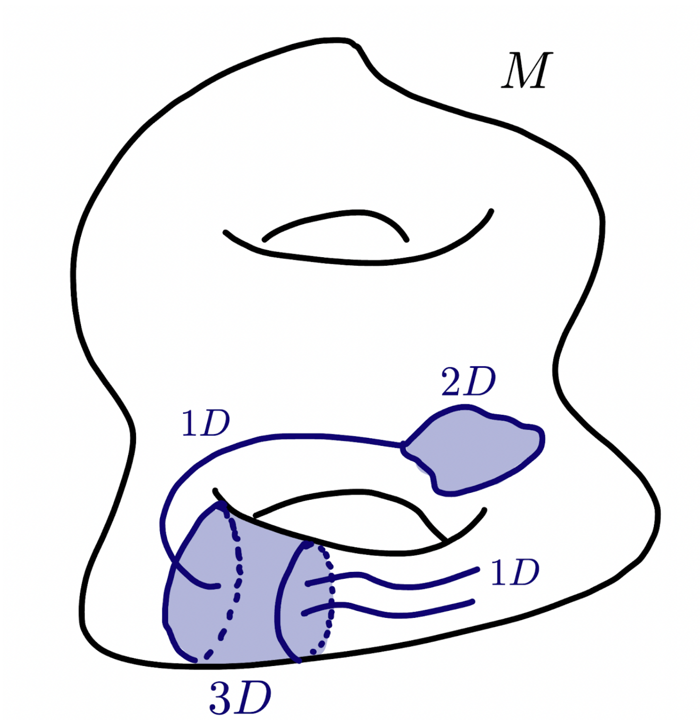

Figure 3: An example of a multivarifold over a general Riemannian manifold M with one, two, and three dimensional strata.

dimension $k(n-k)$ and is called a *Grassmannian* or *Grassmann Manifold*. A detailed construction of the smooth manifold $\Gamma_k(n)$ can be found in [2]. We will also want to consider the manifold

$$G_k(n) = \bigsqcup_{0 \le i \le k} \Gamma_k(n)$$

which consists of all sub-spaces of $\mathbb{R}^n$ of dimension at most k.

Definition 2.3. (Grassmanian Bundles over a Manifold) Let M be an n-dimensional smooth manifold. We denote the tangent bundle of M as TM and similarly denote by $\Gamma_k M$ the bundle of tangent Grassmannians, i.e. at each point $p \in M$ the fiber $\pi^{-1}(p)$ is the set of k-dimensional sub-spaces of T_pM. We can likewise define G_kM to be the bundle whose fibers are $G_k(T_pM)$.

We are now ready to state the definition of a multi-varifold.

Definition 2.4. (Multi-Varifold (Stratified Varifold)) Let M be an n-dimensional Riemannian manifold. A *multivarifold* on M of order k is a Radon measure on G_kM with compact support. Since every Radon measure defines a linear functional on $C_0(G_kM)$ and vice-versa, we can consider the vector space of order k-multivarifolds on M which we will denote V_kM.

Definition 2.5. Let (X, Σ, μ) be a locally compact measure space with μ a Radon measure and let Y be a locally compact space with $h : X \to Y$ measurable, i.e. for all open $U \subset Y$, $h^{-1}(U) \in \Sigma$. Then, we can define the *pushforward measure* $h_\# \mu$ on the Borel subsets of Y via

$$h_\# \mu(E) = \mu(h^{-1}(E)).$$

In particular, if we think of μ as a linear functional on $C_0(X)$, then $h_\# \mu$ is the functional on $C_0(Y)$ given by

$$h_\# \mu(f) = \mu(f \circ h).$$

Definition 2.6. Suppose V is a multivarifold of order k on a manifold M and let $\pi : G_kM \to M$ denote the projection for the bundle G_kM. Then, V induces a measure $\|V\|$ on M via the push-forward $\|V\| := \pi_\# V$. In particular $\|V\|$ has compact support in M.

We will now provide an example which illustrates how multi-varifolds are a generalization of the notion of 'lower dimensional compact sub-manifold.' Let M be an n-dimensional Riemannian manifold and let $N \subset M$ be a compact sub-manifold of dimension k. Let $S : N \to G_kM$ be the map which sends a point $p \in N$ to its k-dimensional tangent space $T_pN \in G_kM$. We can define the multi-varifold $[N]$ on M via its action on functions $f \in C_0(G_kM)$. Let $f \in C_0(G_kM)$. Then, $f \circ S : N \to \mathbb{R}$ so we can define

$$[N](f) = \int_N (f \circ S) d\text{Vol}$$

where dVol is the volume form on N. We can see what measure $[N]$ induces on M by choosing a function $g \in C(M)$ and seeing how $\|[N]\|$ acts on g. By definition,

$$\|[N]\|(g) = (\pi_\#[N])(g) = [N](g \circ \pi) = \int_N (g \circ \pi \circ S) dVol.$$

Since $\pi \circ S = \text{Id}_M$, it follows that the measure $\|[N]\|$ on M is just the k-dimensional volume measure of N. We can therefore see that compact lower dimensional sub-manifolds are a special case of multi-varifolds.

Proposition 2.7. *For $0 \leq i \leq k$, let $V_{k,i}M$ be the vector subspace of V_kM which consists of all multivarifolds supported on Γ_iM. Let $\mathcal{M}(\Gamma_iM)$ be the vector space of compactly supported Radon measures on Γ_iM. Then, $V_{k,i}M$ is canonically isomorphic to $\mathcal{M}(\Gamma_iM)$ via the isomorphism $\rho : \mathcal{M}(\Gamma_iM) \to V_{k,i}M$ which sends a measure μ on Γ_iM to a measure on G_kM such that for all $f \in C(G_kM)$,*

$$\rho(\mu)(f) = \mu(f|_{\Gamma_iM}).$$

The measure $\rho(\mu)$ is by definition supported on Γ_iM and thus belongs to the vector space $V_{k,i}M$.

We now present a key theorem about the structure of multivarifolds.

Theorem 2.8. *(Structure Theorem) If M is an n-dimensional Riemannian manifold and $0 \leq k \leq n$, then we can decompose the vector space V_kM as the direct sum*

$$V_kM = \bigoplus_{0 \leq i \leq k} V_{k,i}M.$$

Detailed proofs for Proposition 2.7 and Theorem 2.8 can be found in [4] and in [3].

We can now write any multivarifold V as a sum of components

$$V = V^0 + V^1 + \cdots + V^k$$

where each $V^i \in V_{k,i}M$. We call the i-dimensional piece of V, V^i, the i-dimensional stratum of V. This decomposition into stratum of each dimension motivates the following definition of the multi-mass of a varifold. Before doing so we recall a standard norm on the vector space of Radon measures.

Definition 2.9. Let X be a compact or locally compact space and let $\mathcal{M}(X)$ be the vector space of Radon measures on X. We identify Radon measures on X with functionals on $C_0(X)$ which are continuous with respect to the topology of uniform convergence on compact sets. We define the norm n on $\mathcal{M}(X)$ to be

$$n(\mu) = \sup_{f \in C_0(X)} \{|\mu(f)|; \|f\|_\infty \leq 1\}.$$

This definition is similar to the standard definition of the operator norm.

Definition 2.10. (Multimass and Principal Mass) Let V be a multivarifold of order k on an n-dimensional Riemannian manifold M. We define the i-dimensional mass of V to be

$$M_i(V) = n(\|V^i\|)$$

where $\|V^i\|$ is the measure on M induced by V^i via the construction in Definition 2.6. Explicitly, if we let $\pi : G_k M \to M$ be the projection mapping, then,

$$M_i(V) = \sup_{f \in C_0(M)} \{|V^i(f \circ \pi)|; \|f\|_\infty \leq 1\}.$$

We call the tuple $(M_1(V), \ldots, M_k(V))$ the *multimass* of V and the number $M_k(V)$ the *principal mass* of V.

2.2 Maps and Induced Maps

In this section we will discuss an important construction which allows one to lift a smooth map of manifolds $f : M \to N$ to a map of multivarifolds $T_f : W \to V$ where $W \in V_k M$ and $V \in V_k N$. This is a crucial construction in our quest to understand the Plateau problem since it allows us to parameterize multivarifolds.

Definition 2.11. Let X be a locally compact space and μ a Radon measure. Let f be a bounded measurable function. Then, we can define a new Radon measure ν via

$$\nu(E) = \int_E f d\mu$$

for all $E \in \Sigma$. If we take the functional analysis perspective and think of μ as a functional on $C_0(X)$, then ν is the functional

$$g \mapsto \mu(f \cdot g).$$

We will adopt the notation of [4] and denote ν as $\mu \wedge f$.

Definition 2.12. Let M, N be Riemannian manifolds and $f : M \to N$ a map which is at least C^1. Let $\dim(M) = m$ and $\dim(N) = n$. Fix a point $x \in M$. Then the differential of f, $Df(x) : T_x M \to T_{f(x)} N$ sends linear subspaces of $T_x M$ to linear subspaces of $T_{f(x)} N$. Therefore, we get an induced map $G(f) : G_m M \to G_n N$ defined by

$$(x, W) \mapsto (f(x), Df(x)(W)).$$

Define a map $\tau(f) : G_m M \to \mathbb{R}$ by

$$\tau(f)(x, W) = |\det(Df(x)|_W)|.$$

We can now define the *induced map of type* T to be the map $T_f : G_k M \to G_k N$ given by the formula

$$T_f(V) = G(f)_\#(V \wedge \tau(f)).$$

Intuitively, we can think of this is a type of change of variables for multivarifolds. If we denote $V(g)$ as $\int g dV$ and $T_f V(h)$ as $\int h d(T_f V)$ then we get the more familiar looking formula

$$\int h d(T_f V) = (V \wedge \tau(f))(h \circ G(f)) = \int_{(x,W) \in G_k M} h(f(x), Df(x)(W)) |\det Df(x)|_W| dV.$$

2.3 Topologies on the Space $V_k M$

We will now introduce several topologies on the vector space $V_k M$. Our ultimate goal is to prove the existence of a certain multivarifold which, among a certain class of multivarifolds, has minimal principal mass. In order to achieve this goal we will construct the *parametric topology*, show that the principal mass is lower semi-continuous with respect to this topology, and then restrict to looking at compact subsets.

2.3.1 Weak and Strong Topologies

Definition 2.13. (Strong Topology) We will refer to the topology on V_kM induced by the norm n as the *strong topology* on V_kM.

Definition 2.14. (Weak Topology) The *weak topology* on V_kM is the weakest topology such that the mapping $\mu \mapsto \int f d\mu$ is continuous for all $f \in C_0(G_kM)$. Equivalently, the weak topology is the weakest topology which guarantees that a sequence $(\mu_n)_{n\in\mathbb{N}}$ converges to a measure μ if for all $f \in C_0(G_kM)$,

$$\int f d\mu_n \to \int f d\mu.$$

We now prove a useful lemma about lower semi-continuous functions which allows us to shows that the multimass is lower semi-continous with respect to the weak topology.

Lemma 2.15. *Let X be a topological space and let $\{f_l\}_{l\in\lambda}$ be a family of extended real valued functions on X. Define the function $f : X \to \overline{\mathbb{R}}$ by*

$$f(x) = \sup_{l\in\lambda} f_\lambda(x).$$

Then, f is lower semi-continuous on X.

Proof. In order to show that f is lower semi-continuous it suffices to show that for all $p \in \overline{\mathbb{R}}$, $f^{-1}(p, \infty] \subset X$ is open. Fix $p \in \overline{\mathbb{R}}$. Then,

$$f^{-1}(p, \infty] = \{x \in X \mid \sup_l f_l(x) > y\} = \{x \in X \mid \text{ there exists a } l \in \lambda \text{ such that} f_l(x) > p\}.$$

Therefore,

$$f^{-1}(p, \infty] = \bigcup_{l\in\lambda} \{x \in X \mid f_\lambda(x) > p\} = \bigcup_{l\in\lambda} f_l^{-1}(p, \infty].$$

Since each f_l is continuous, for every $l \in \lambda$ the set $f_l^{-1}(p, \infty]$ is open. Thus, $f^{-1}(p, \infty]$ is the union of open sets and is therefore open. Since $p \in \overline{\mathbb{R}}$ was arbitrary, f is lower semi-continuous. □

Theorem 2.16. *The i-dimensional mass functions $M_i(V)$ are lower semi-continuous with respect to the weak topology.*

Proof. Let $V \in V_kM$ and fix $i \leq k$. Let χ_i be a continuous functions such that $\chi_i(x, W) = 1$ when $\dim(W) = i$ and $\chi_i(x, W) = 0$ otherwise. Then,

$$M_i(V) = \sup_{f\in C_0(M)} \{|V^i(f \circ \pi)|; \|f\|_\infty \leq 1\} = \sup_{f\in C_0(M)} \{|V(\chi_i \cdot (f \circ \pi))|; \|f\|_\infty \leq 1\}.$$

By Definition 2.14, each mapping $H^i_f : V_kM \to \mathbb{R}$ given by

$$H^i_f(V) = |V(\chi_i \cdot (f \circ \pi))|$$

is continuous with respect to the weak topology. Therefore,

$$M_i(V) = \sup_{f\in C_0(G_kM), \|f\|_\infty \leq 1} H^i_f(V).$$

It thus follows from Lemma 2.15 that $M_i(V)$ is lower semi-continuous with respect to the weak topology. □

The fact that the mass functions are lower-semi-continuous in the weak topology will be a crucial step in our final proof that the principal mass is lower semi-continuous with respect to the parametric topology.

2.3.2 Parametrizations and the Parametric Topology

Definition 2.17. (Parametrization of a Multivarifold) Let M, N be manifolds and let $V \in V_k M$. A C^k parametrization of V is a pair (W, f) where $W \in V_k N$ and $f : N \to M$ is a C^k mapping such that $V = T_f W$.

Definition 2.18 (T-Equivalence)**.** Let M, N be manifolds and let $V \in V_k M$. Let f_1, f_2 be C^k mappings from N to M and let $W \in V_k N$. Then, we say that the pairs (W, f_1) and (W, f_2) are *T-equivalent* if $T_{f_1} W = T_{f_2} W$.

From here on out we will primarily be concerned with multivarifolds on Euclidean spaces rather than arbitrary Riemannian Manifolds, however all of the following results do carry over to the more general case.

Definition 2.19. Let W be a multivarifold on $\mathbb{R}^n$. Let $\mathcal{P}^k(W, \mathbb{R}^m)$ denote the set of multivarifolds on $\mathbb{R}^m$ which can be parametrized by pairs (W, f) where $f : \mathbb{R}^n \to \mathbb{R}^m$ is a C^k mapping. We can identify $\mathcal{P}^k(W, \mathbb{R}^m)$ as being the set of equivalence classes of C^k functions under the relation of being $T-$equivalent as defined in Definition 2.17.

Definition 2.20. Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be of class C^k. Then, for each set of indices of the form $\alpha = (\alpha_1, \ldots, \alpha_n)$ where $\alpha_i \in \mathbb{N}$ and $\sum_{i=1}^n \alpha_i = k$, we denote the partial derivative

$$\frac{\partial^{(\alpha_1 + \cdots + \alpha_n)} f}{\partial x_1^{\alpha_1} \ldots \partial x_n^{\alpha_n}}$$

as $D^\alpha f$. Note that we allow for the case $D^0 f = f$. For a fixed $\alpha = (\alpha_1, \ldots, \alpha_n)$, we define $|\alpha| = \alpha_1 + \cdots + \alpha_n$.

Definition 2.21. Let X and Y be metric spaces. We say that a function $f : X \to Y$ is *locally Lipschitz* with constant $Lip(f) \leq L$ if for every $x \in X$ there exists an open ball $B(x, r)$ such that $f|_{B(x,r)}$ is Lipschitz with Lipschitz constant L.

Definition 2.22. (Parametric Topology) Let $\mathcal{P}^k(W, \mathbb{R}^m)$ be defined as in Definition 2.19. Let $\{V_n\}_{n \in \mathbb{N}}$ be a sequence in $\mathcal{P}^k(W, \mathbb{R}^m)$ and let $\{f_n\}_{n \in \mathbb{N}} \subset C^k(\mathbb{R}^n, \mathbb{R}^m)$ be the sequence of functions such that $V_n = T_{f_n} W$ for all $n \in \mathbb{N}$. Then, the sequence $(V_n)_{n \in \mathbb{N}}$ converges to the multivarifold $V = T_f W$ in the *parametric topology* if and only if for all α, the sequence $(D^\alpha f_n)_{n \in \mathbb{N}}$ converges to $D^\alpha f$ locally uniformly. Note that the parametric topology can also be defined via a metric. This is the approach adopted by [4].

Lemma 2.23. *Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be locally Lipschitz with $Lip(f) \leq M$. Let $C \subset \mathbb{R}^n$ be a convex set. Then, f is Lipschitz on C with constant M.*

Proof. Let $x, y \in C$ and let l be the closed line segment connecting x to y. For every $t \in L$, let $B(r, r_t)$ be a ball centered at t on which f is Lipschitz with constant less than or equal to M. Since l is compact, we can cover l by a finite collection of balls of the form $B(t, \frac{r_t}{2})$. Let $\mathcal{G} = \{B(t_i, \frac{r_i}{2})\}_{i=1}^k$ denote such a finite covering where the $t_i' s$ are indexed in increasing order along l, so that t_1 is closest to x and t_k is closest to y. Since l is connected,

$$B(t_i, \frac{r_i}{2}) \cap B(t_{i+1}, \frac{r_{i+1}}{2}) \neq \emptyset$$

for all $1 \leq i < k$. For every $1 \leq i < k$, let $t_{i,i+1}$ be an element of $B(t_i, \frac{r_i}{2}) \cap B(t_{i+1}, \frac{r_{i+1}}{2})$ such that $t_{i,i+1} \neq t_j$ for all $1 \leq j \leq k$. Note that the finite sequence $x, t_1, t_{1,2} \ldots, t_{k-1,k}, t_k, y$ is strictly increasing in the linear order on l. Therefore,

$$\begin{aligned}
\|f(x) - f(y)\| &\leq \|f(x) - f(t_1)\| + \|f(t_1) - f(t_{1,2})\| + \|f(t_{12}) - f(t_2)\| + \cdots + \|f(t_{k-1,k}), f(t_k)\| + \|f(t_k) - y\| \\
&\leq M\|x - t_1\| + M\|t_1 - t_{1,2}\| + M\|t_{1,2} - t_2\| + \cdots + M\|t_{k-1,k} - t_k\| + M\|t_k, y\| \\
&= M\|y - x\|.
\end{aligned}$$

Since $x, y \in C$ were arbitrary we have proven that $f|_C$ is Lipschitz with constant M. □

Theorem 2.24. *Suppose $W \subset \mathbb{R}^n$ is a compact sub-manifold which is at least C^1. For $\kappa > 0$, define*

$$A_\kappa = \{V = T_f W \in \mathcal{P}^k(W, \mathbb{R}^m) \mid D^\alpha f : \mathbb{R}^n \to \mathbb{R}^m \text{ is locally Lipschitz with } Lip(D^\alpha f) \leq \kappa \text{ for all } \alpha\}.$$

Then, for all $\kappa > 0$, the set A_κ is compact with respect to the parametric topology.

Proof. Fix $\kappa > 0$ and let $(V_n)_{n\in\mathbb{N}}$ be a sequence in A_κ. Let $(f_n)_{n\in\mathbb{N}}$ be the sequence of functions such that $V_n = T_{f_n}W$ for all $n \in \mathbb{N}$. Since $W \subset \mathbb{R}^n$ is compact, it is contained in the interior of some large, but bounded, closed ball B. In particular, because $V_n = T_{f_n}W$ only depends on the behavior of f in a neighborhood of W, we can assume that every f_n is supported in $int(B)$. We can do this because in $\mathcal{P}^k(W, \mathbb{R}^m)$, we identify functions which are T-equivalent.

Since B is convex, we can apply Lemma 2.23 and get that for every $k \in \mathbb{N}$ and $\alpha = (\alpha_1, \ldots, \alpha_n)$, $D^\alpha f$ is Lipschitz on B with constant κ. Fix $\alpha = (\alpha_1, \ldots, \alpha_n)$ where each $\alpha_i \geq 0$. We will show that $(D^\alpha f_k)_{k\in\mathbb{N}}$ has a subsequence which converges uniformly B and then extend to saying that (f_n) has a subsequence which converges in the parametric topology using a diagonal argument.

In order to show that $(D^\alpha f_k)_{k\in\mathbb{N}}$ has a subsequence which converges uniformly on B, we will utilize the compactness of B to apply the Arzela-Ascoli Theorem. We first show that the family $\{D^\alpha f_k\}_{k\in\mathbb{N}}$ is uniformly bounded. Without loss of generality, suppose that B is a closed ball centered at the origin of radius R. Fix an $i \in \mathbb{N}$. Then, for all $x \in B$,

$$\|D^\alpha f_i(x)\| \leq \kappa\|x\| \leq \kappa R.$$

Therefore, $\{D^\alpha f_k\}_{k\in\mathbb{N}}$ is uniformly bounded with bound κR. We will now show that $\{D^\alpha f_k\}_{k\in\mathbb{N}}$ is an equicontinuous family of functions.

Let $b \in B$ and fix $\varepsilon > 0$. Let $\delta = \varepsilon/\kappa$. Let $b' \in B$ such that $\|b - b'\| < \delta$. Then, for every $i \in \mathbb{N}$,

$$\|D^\alpha f_i(b) - D^\alpha f_i(b')\| \leq \kappa\|b - b'\| < \kappa\delta < \varepsilon.$$

Therefore, $\{D^\alpha f_k\}_{k\in\mathbb{N}}$ is both equicontinuous and uniformly bounded. Since B is compact, we can apply the Arzela-Ascoli Theorem to get that there exists a subsequence $\{D^\alpha f_{k_i}\}$ which converges uniformly on B.

Because the previous argument only utilizes the Lipschitz properties of $D^\alpha f_i$, we have shown that any subsequence of $(D^\alpha f_i)$ also has a subsequence which converges uniformly on B. Let $\alpha^1, \alpha^2, \ldots, \alpha^J$ denote an enumeration of every possible tuple α with $|\alpha| \leq k$. Since there are finitely many α^j's, we can find a set of indices $\{k_i\}$ such that $(D^{\alpha^j} f_{k_i})$ converges uniformly on B for all $1 \leq j \leq J$. Let

$$f = \lim_{i\to\infty} f_{k,i}.$$

Let $V = T_f W$. Since each $D^{\alpha^j} f$ converges uniformly on B which contains $\text{supp} D^{\alpha^j} f$, $D^{\alpha^j} f$ converges locally uniformly for all $1 \leq j \leq J$. Therefore, in the parametric topology

$$\lim_{i\to\infty} V_{k_i} = V.$$

Since we have extracted a convergent subsequence from an arbitrary sequence in A_κ, to show compactness we need only show that $V \in A_\kappa$. Since $(f_{k_i})_{i\in\mathbb{N}}$ converges uniformly on the open set $int(B)$ and so do all of its partial derivatives, $f|_{int(B)}$ is a C^k function satisfying

$$D^{\alpha_j} f := \lim_{i\to\infty} D^{\alpha^j} f_{k_i}$$

for all $1 \leq j \leq J$. Because $int(B)$ contains the support every $D^{\alpha^j} f_i$, f is a C^k function. We now need to show that for all α^j, $D^{\alpha^j} f$ is still locally Lipschitz with a constant less than or equal to κ. Fix α^j and $\varepsilon > 0$. Let k_i be sufficiently large so that

$$\sup_{y\in B}\|D^{\alpha^j} f(y) - D^{\alpha^j} f_{k_i}(y)\| < \frac{\varepsilon}{2}.$$

We have previously shown that $D^{\alpha^j} f_{k_i}$ is Lipschitz with a constant at most κ on B. Let $a, b \in B$. Then,

$$\begin{aligned}\|D^{\alpha^j} f(a) - D^{\alpha^j} f(b)\| &\leq \|D^{\alpha^j} f(a) - D^{\alpha^j} f_{k_i}(a)\| + \|D^{\alpha^j} f_{k_i}(a) - D^{\alpha^j} f_{k_i}(b)\| + \|D^{\alpha^j} f_{k_i}(b) - b\| \\ &\leq \varepsilon + \kappa\|a - b\|.\end{aligned}$$

Since $\varepsilon > 0$ was arbitrary,

$$\|D^{\alpha^j} f(a) - D^{\alpha^j} f(b)\| \leq \kappa\|a - b\|$$

for all $a, b \in B$. This shows that $D^{\alpha^j} f$ is Lipschitz on B with constant κ which immediately implies that it is locally Lipschitz with the same constant because $\text{supp}(D^{\alpha^j} f_{k_i}) \subset int(B)$. We have thus shown that every sequence $(V_n) \subset A_\kappa$ has a subsequence which converges in the parametric topology. Therefore, A_κ is compact in this topology. □

3 Solving a Generalized Version of Plateau's Problem

We will now show that the principal mass functions is lower semi-continuous with respect to the parametric topology. It's important to note that only the principal mass function is lower semi-continuous in general and the other lower dimensional mass functions may not be. This is because higher dimensional strata can be continuously crushed into lower dimensional pieces which will suddenly add to the lower dimensional mass. However, lower dimensional strata cannot deform into higher dimensional strata and discontinuously add to the higher dimensional volume.

Theorem 3.1. *The principal mass function is lower semi-continuous with respect to the parametric topology.*

Proof. Fix a multivarifold W of order k over $\mathbb{R}^n$. We want to show that the map $T_f W \mapsto M_k(T_f W)$ is lower semi-continuous with respect to the parametric topology on $\mathcal{P}^k(W, \mathbb{R}^M)$. Suppose that the sequence $(V_n)_{n\in\mathbb{N}}$ converges to a multivarifold V in the parametric topology. Let (f_n) be such that $V_n = T_{f_n} W$.

We will show that $V_n^k \to V^k$ in the weak topology. Fix an arbitrary function $g \in C_0(G_k\mathbb{R}^m)$. Then, for each $n \in \mathbb{N}$,

$$V_n^k(g) = V_n(\chi_k \cdot g) = \int_{(x,A)\in G_k\mathbb{R}^n} \tau(f_n) \cdot (\chi_k g)(f_n(x), Df_n(x)(A))dW.$$

For every $n \in \mathbb{N}$, define $h_n : G_k\mathbb{R}^n \to \mathbb{R}$ by

$$h_n(x, A) = \tau(f_n)(x, A) \cdot (\chi_k g)(f_n(x), Df_n(x)(A)) = |\det(Df_n(x)|_A)| \cdot (\chi_k g)(f_n(x), Df_n(x)(A)).$$

Let K be a compact subset of $G_k\mathbb{R}^n$ and let $K' \subset \mathbb{R}^n$ denote $\pi(K)$. Then, $f_n \to f$ and $Df_n \to f$ uniformly on $\pi(K)$. Since $\chi_k g$ and the absolute value of the determinant are continuous functions, h_n converges uniformly to

$$h(x, A) = |\det(Df(x)|_A)| \cdot (\chi_k g)(f(x), Df(x)(A))$$

on K. Since K was arbitrary and W is continuous with respect to the topology of uniform convergence on compact subsets,

$$W(h_n) \to W(h).$$

Since $W(h_n) = V_n^k(g)$ and $W(h) = V^k(g)$, we get that

$$V_n^k(g) \to V_k(g).$$

Since $g \in C_0(\mathbb{R}^m)$ was arbitrary, $V_n^k \to V^k$ in the weak topology. Therefore, by Theorem 2.16,

$$M_k(V_n) \to M_k(V).$$

Thus, the principal mass function is lower semi-continuous with respect to the parametric topology. □

Theorem 3.2. *(Solution to a Multivarifold Version of Plateau's Problem) Let $B \subset \mathbb{R}^m$ be a compact sub-manifold of dimension $k-1$ and let $W \subset \mathbb{R}^n$ be compact sub-manifold of dimension k such that ∂W is homeomorphic B. Let H be the set of C^s $(s \geq 1)$ mappings $f : \mathbb{R}^n \to \mathbb{R}^m$ such that $f|_{\partial W}$ is a homeomorphism of ∂W onto B. Then, for every $\kappa > 0$ there exists a function f in the set*

$$H \cap A_\kappa$$

which has the least principal mass.

Proof. Fix a positive real number $\kappa > 0$. By Theorem 2.24 the set A_κ is compact in the parametric topology. Since H is closed in the parametric topology, $H \cap A_\kappa$ is compact. Therefore, $H \cap A_\kappa$ is compact in the parametric topology. Because the principle mass function is lower semi-continuous and $H \cap A_\kappa$ is compact, there will be a parametrization $f \in H \cap A_\kappa$ such that $T_f W$ has minimal principal mass. ☐

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