DYNAMICS IN THE PLANE AND THE POINCARÉ-BENDIXSON THEOREM

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ABSTRACT. In this paper we will discuss smooth dynamical systems in $\mathbb{R}^2$ and prove the Poincaré-Bendixson Theorem. The Poincaré-Bendixson Theorem is a powerful and fundamental result which, under suitable conditions, fully characterizes the long term behavior of smooth dynamical systems in the plane. We will also present an application of the Poincaré-Bendixson theorem to a system of differential equations which models the excitability of a neuron. Lastly, we will use the Poincaré-Bendixson Theorem to prove the Brouwer-Fixed Point Theorem for convex subsets of $\mathbb{R}^2$.

CONTENTS

1. INTRODUCTION

Non-linear differential equations are needed to model phenomena in almost every quantitative field, such as physics, chemistry, biology, epidemiology, and engineering. Although most non-linear dynamical systems can not be solved analytically, there are a host of known techniques which can be used to obtain information about what solutions look like. One particularly powerful technique for analyzing the long term limiting behavior of a smooth dynamical system is Poincaré-Bendixson Theory. The Poincaré-Bendixson Theorem alongside its related corollaries are especially useful for proving the existence of periodic solutions which would be difficult to find as explicit solutions to a system of differential equations. Furthermore, the Poincaré-Bendixson Theorem rules out the possibility of chaos for smooth two dimensional systems. The key ideas of Poincaré-Bendixson Theory were introduced by Poincaré in the 1880's, however the Poincaré-Bendixson Theorem was not fully fleshed out and rigorously justified until it was proved by Ivar Bendixson in 1901 [1].

2. PRELIMINARIES

Throughout this paper we will assume that the reader has some familiarity with the basics of topology and analysis in $\mathbb{R}^n$. Although many of the following definitions and lemmas are stated about $\mathbb{R}^n$, we will be primarily concerned with planar systems defined in $\mathbb{R}^2$. Capital letters will be used to denote vectors and matrices while lower case letters will be used to denote real numbers.

Definition 2.1. A continuously differentiable function $\varphi : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ is a *smooth dynamical system* if it satisfies the following conditions:

(1) For all $X \in \mathbb{R}^n$, $\varphi(0, X) = X$.
(2) For all $X \in \mathbb{R}^n$ and $t, s \in \mathbb{R}$, $\varphi(t, \varphi(s, X)) = \varphi(t + s, X)$.

We will adopt the conventional notation of writing $\varphi(t, X)$ as $\varphi_t(X)$. The function φ can be thought of as outputting where in $\mathbb{R}^n$ the dynamical system goes after a time t starting with an initial condition X. The first condition states that if no time passes, then the system remains at the initial position X. The second condition placed on φ says that starting the system at X and letting it run $t+s$ units of time is the same as first running it s units of time from X and then t units of time starting from $\varphi(s, X)$.

We will be concerned with smooth dynamical systems as solutions to systems of autonomous first order differential equations in the plane. These are differential equations of the form

$$X' = F(X),$$

where $F : \mathbb{R}^2 \to \mathbb{R}^2$ is a time independent C^1 vector field.

Solutions to these types of systems can be thought of as curves in $\mathbb{R}^2$ which are always tangent to the vector field F. More precisely, they are functions $X : I \to \mathbb{R}$, where $I \subset \mathbb{R}$ is an interval, such that $\frac{d}{dt}X(t) = F(X(t))$ for all $t \in I$. Whether such solutions exist or are unique is summarized in the following fundamental theorem.

Theorem 2.2 (Existence and Uniqueness)**.** *Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be continuously differentiable. Consider the initial value problem $X' = F(X)$ and $X(0) = X_0$, where $X_0 \in \mathbb{R}^n$. Then, there exists a real number $a > 0$ and a unique function $X : (-a, a) \to \mathbb{R}^n$ which solves this initial value problem.*

Proof. A detailed and thorough treatment of this theorem can be found in Chapter 17 of [2]. □

Given a C^1 autonomous system $X' = F(X)$, the associated smooth dynamical system is the function $\varphi_t(Y)$ which outputs the solution to the initial value problem $X(0) = Y$, evaluated at time t. We will also refer to $\varphi_t(Y)$ as the solution of the autonomous system through Y. The general function $\varphi_t(Y)$, which depends on the initial condition Y, is also called the *flow* of the system $X' = F(X)$.

Theorem 2.3 (Smoothness of Flows)**.** *Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be a C^1 vector field. Let $\varphi : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ be the flow of the autonomous system $X' = F(X)$. Then, $\varphi_t(X)$ is C^1, i.e. $\frac{\partial \varphi}{\partial X}$ and $\frac{\partial \varphi}{\partial t}$ exist and are continuous.*

Proof. A full proof of this theorem can be found on page 402 of [2]. □

The smoothness of flows is important because it means that flows of C^1 autonomous systems of differential equations are smooth dynamical systems according to Definition 1.1. It also guarantees that φ is continuous with respect to initial conditions.

Definition 2.4. An *equilibrium point* of the autonomous system $X' = F(X)$ is a point $Y \in \mathbb{R}^n$ for which $F(Y) = 0$.

Definition 2.5. Suppose $Y \in \mathbb{R}^n$ is not an equilibrium point of the autonomous system $X' = F(X)$. If there exists a strictly positive real number τ such that $\varphi_\tau(Y) = Y$, then $\varphi_t(Y)$ is called a *periodic solution* or *closed orbit*. The smallest $\tau > 0$ for which $\varphi_\tau(Y) = Y$ is the *period* of the closed orbit.

It is worth noting that for a closed orbit, by Definition 1.1,

$$\begin{aligned} \varphi_{\tau+t}(Y) &= \varphi_t(\varphi_\tau(Y)) \\ &= \varphi_t(Y) \end{aligned}$$

for any $t \in \mathbb{R}$.

Definition 2.6. A set $A \subset \mathbb{R}^n$ is *positively invariant* if for all $X \in A$, $\varphi_t(X) \in A$ for all $t > 0$. Similarly, A is *negatively invariant* if for all $X \in A$, $\varphi_t(X) \in A$ for all $t < 0$. A set A is *invariant* if for all $X \in A$, $\varphi_t(X) \in A$ for all $t \in \mathbb{R}$.

Definition 2.7. The *ω-limit set* of $\varphi_t(X)$ is the set of points $Y \in \mathbb{R}^n$ for which there exists a strictly increasing sequence $(t_n)_{n=0}^{\infty}$, such that $\lim_{n\to\infty} t_n = \infty$ and $\lim_{n\to\infty} \varphi_{t_n}(X) = Y$. We often denote the ω-limit set of the solution through X as just $\omega(X)$.

The *α-limit set* of $\varphi_t(X)$ is the set of points $Y \in \mathbb{R}^n$ for which there exists a strictly decreasing sequence $(t_n)_{n=0}^{\infty}$, such that $\lim_{n\to\infty} t_n = -\infty$ and $\lim_{n\to\infty} \varphi_{t_n}(X) = Y$. Similarly, we will denote the α-limit set of the solution through X as just $\alpha(X)$.

Remark 2.8. All periodic orbits are invariant. Furthermore, if γ is a closed orbit and $X \in \gamma$, then $\omega(X)$ and $\alpha(X)$ are equal to γ.

We will now prove some useful lemmas about α and ω-limit sets. For the following lemmas we will be assuming that $\varphi_t(X)$ is the flow of C^1 autonomous system in $\mathbb{R}^n$.

Lemma 2.9. *Suppose that Y and Z both lie on the same solution curve of the system $X' = F(X)$. Then, $\omega(Z) = \omega(Y)$ and $\alpha(Z) = \alpha(Y)$.*

Proof. Suppose that Y and Z both lie on the same solution curve of the system $X' = F(X)$. Let $X \in \omega(Y)$. Then, there exists a strictly increasing divergent sequence $(t_n)_{n=1}^{\infty}$ such that $\lim_{n\to\infty} \varphi_{t_n}(Y) = X$. Because Y and Z are on the same solution curve, there exists a real number s such that $\varphi_s(Y) = Z$. Moreover, there exists a natural number m such that $t_m > s$. For all $k \geq m$,

$$\begin{aligned} \varphi_{t_k}(Y) &= \varphi_{(t_k - s + s)}(Y) \\ &= \varphi_{(t_k - s)}(\varphi_s(Y)) \\ &= \varphi_{(t_k - s)}(Z). \end{aligned}$$

Therefore,

$$\begin{aligned} \lim_{n\to\infty} \varphi_{(t_n - s)}(Z) &= \lim_{n\to\infty} \varphi_{t_n}(Y) \\ &= X. \end{aligned}$$

Thus, $\omega(Y) \subset \omega(Z)$. The same reasoning can be used to show that $\omega(Z) \subset \omega(Y)$. Therefore, $\omega(Y) = \omega(Z)$. Similar reasoning can be used to show that $\alpha(Y) = \alpha(Z)$. ☐

Lemma 2.10. *For all $X \in \mathbb{R}^n$, $\omega(X)$ and $\alpha(X)$ are invariant sets.*

Proof. Let $Y \in \omega(X)$. Let (t_n) be an increasing divergent sequence such that $\lim_{n\to\infty} \varphi_{t_n}(X) = Y$. Let $s \in \mathbb{R}$ be arbitrary. By Theorem 2.3, φ is continuous in all of its variables. Therefore,

$$\begin{aligned} \lim_{n\to\infty} \varphi_{(s+t_n)}(X) &= \lim_{n\to\infty} \varphi_s(\varphi_{t_n}(X)) \\ &= \varphi_s(\lim_{n\to\infty} \varphi_{t_n}(X)) \\ &= \varphi_s(Y). \end{aligned}$$

Since $(s + t_n)$ is strictly increasing and divergent, by Definition 2.7, $\varphi_s(Y) \in \omega(X)$. Because s was arbitrary, $\omega(X)$ is invariant by Definition 2.6. ☐

Lemma 2.11. *The sets $\omega(X)$ and $\alpha(X)$ are closed.*

Proof. Let $Y \in \overline{\omega(X)}$. Let $B(r, Y)$ denote the open ball of radius r centered at Y. For every $n \in \mathbb{N}$, $B(\frac{1}{n}, Y) \cap \omega(X)$ is nonempty. We can then find a sequence, (Y_n), in $\omega(X)$ such that for all $n \in \mathbb{N}$, $\|Y_n - Y\| < \frac{1}{n}$. Because (Y_n) is in $\omega(X)$, by Definition 2.7, for every $m \in \mathbb{N}$ there exists an arbitrarily large $t_m \in \mathbb{R}$ such that $\|\varphi_{t_m}(X) - Y_m\| < \frac{1}{m}$. We can then construct a sequence (t_k) such that for all $k \in \mathbb{N}$, $\|\varphi_{t_k}(X) - Y_k\| < \frac{1}{k}$ and $t_{k+1} > t_k + 1$.

Thus, for all k,

$$\|Y - \varphi_{t_k}(X)\| \leq \|Y - Y_k\| + \|Y_k - \varphi_{t_k}(X)\| \leq \frac{2}{k}.$$

We have thus found an increasing divergent sequence (t_k) such that $\lim_{k\to\infty} \varphi_{t_k}(X) = Y$. Therefore, by Definition 2.7, $Y \in \omega(X)$. Thus, $\omega(X)$ is closed. Similar reasoning can be used to show that $\alpha(X)$ is also closed. ☐

Lemma 2.12. *If A is a closed positively invariant set and $X \in A$, then $\omega(X) \subset A$. Similarly, if B is a closed negatively invariant set and $Y \in B$, then $\alpha(Y) \subset B$. Furthermore, if C is a closed invariant set, then C contains the ω and α-limit sets of every point in it.*

Proof. Suppose that $A \subset \mathbb{R}^n$ is closed and positively invariant. Let $X \in A$. Let $P \in \omega(X)$. Then, there exists a strictly increasing divergent sequence $(t_n)_{n=1}^{\infty}$ such that $\lim_{n\to\infty} \varphi_{t_n}(X) = P$. Because $(t_n)_{n=1}^{\infty}$ diverges to positive infinity, there exists a natural number m such that $t_n > 0$ for all $n \geq m$. Since A is positively invariant, $\varphi_{t_n}(X) \in A$ for all $t_n \in (t_n)_{n=m}^{\infty}$. Because only finitely many terms have been removed from the sequence, $(\varphi_{t_n}(X))_{n=m}^{\infty}$ still converges to P. Therefore, since A is closed and $\varphi_{t_n}(X) \in A$ for all $n \geq m$, it follows that $P \in A$. Since P was an arbitrary member of $\omega(X)$, $\omega(X) \subset A$.

A very similar argument with the direction of time reversed can be used to show that if $B \subset \mathbb{R}^n$ is closed and negatively invariant and $Y \in B$, then $\alpha(Y) \subset B$. Because invariant sets are both positively and negatively invariant, both of the previous results hold. Therefore, if C is closed and invariant and $Z \in C$, then $\omega(Z) \subset C$ and $\alpha(Z) \subset C$. □

Lemma 2.13. *The set $\omega(X)$ is compact if, and only if there exists an $s \in \mathbb{R}$ such that the set $\{\varphi_t(X) \mid t \geq s\}$ is bounded. Similarly, $\alpha(X)$ is compact if, and only if there exists an $s \in \mathbb{R}$ such that the set $\{\varphi_t(X) \mid t \leq s\}$ is bounded.*

Proof. Suppose that $X \in \mathbb{R}^n$ such that there exists a real number s such that the set $\{\varphi_t(X) \mid t \geq s\}$ is bounded by $r \in \mathbb{R}$. Then, the closed disk of radius r is positively invariant for the point $\varphi_s(X)$. By Lemma 2.12, $\omega(\varphi_s(X))$ is a subset of the closed disk of radius r. Because $\varphi_s(X)$ is on the same solution curve as X, by Lemma 2.9, $\omega(\varphi_s(X)) = \omega(X)$. Therefore, $\omega(X)$ is bounded. By Lemma 2.11 $\omega(X)$ is closed. Thus, by the Heine-Borel Theorem, $\omega(X)$ is compact.

Suppose now that $Y \in \mathbb{R}^n$ such that $\omega(Y)$ is compact. Then $\omega(Y)$ is bounded. Therefore, there exists an $a \in \mathbb{R}$ such that $\omega(Y)$ is bounded by the open ball of radius a. Let B_a denote this ball of radius a. Suppose, for the sake of contradiction, that for every real number s, the set $\{\varphi_t(Y) \mid t \geq s\}$ is unbounded.

Let $P \in \omega(Y)$. By Definition 2.7, there exists an increasing divergent sequence (t_n) such that $\lim_{n\to\infty} \varphi_{(t_n)}(Y) = P$. Because B_a is an open set containing P, there exist infinitely many $\varphi_{t_n}(Y)$ in B_a. Since $\{\varphi_t(Y) \mid t \geq t_n\}$ is unbounded for all $n \in \mathbb{N}$, $\varphi_t(Y)$ must cross the boundary of B_a infinitely many times (see Figure 1). Let $(\varphi_{a_k}(Y))$ be a sequence in the boundary of B_a such that (a_k) is increasing and divergent. Because the boundary of B_a is bounded and closed, there exists a subsequence of (a_k) which converges to a point $Q \in \partial B_a$. Then, $Q \in \omega(X)$. This is a contradiction since $\omega(Y) \subset B_a$ and $B_a \cap \partial B_a = \emptyset$. We have therefore proven by contradiction that if $\omega(Y)$ is compact, then there exists an $s \in \mathbb{R}$ such that $\{\varphi_t(Y) \mid t \geq s\}$ is bounded. Similar reasoning can be used to show that $\alpha(Y)$ is compact if, and only if there exists an $s \in \mathbb{R}$ such that $\{\varphi_t(Y) \mid t \leq s\}$ is bounded. □

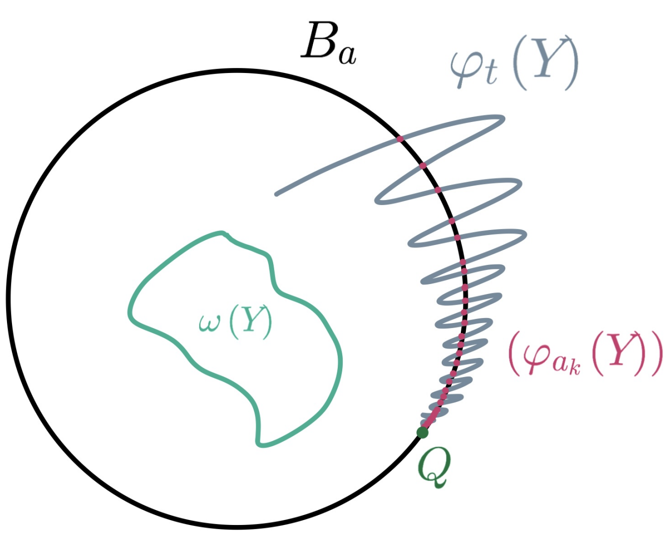

FIGURE 1

We will now prove an important preliminary result about the topology of limit sets.

Lemma 2.14. *If U and V are disjoint nonempty open subsets of $\mathbb{R}^n$, then $\partial U \cap V = \emptyset$ and $\partial V \cap U = \emptyset$.*

Proof. Suppose, for the sake of contradiction, that there exists a $P \in U$ such that $P \in \partial V$. Because U is open, there exists an open neighborhood, N, of P such that $N \subset U$. Since $P \in \partial V$ and N is an open neighborhood containing P, it follows that $N \cap V \neq \emptyset$. This contradicts the fact that U and V are disjoint. The same reasoning can be used to conclude that if $P \in \partial U$, then $P \notin V$.

□

Theorem 2.15. *For all $X \in \mathbb{R}^n$, if $\omega(X)$ is compact, then $\omega(X)$ is nonempty and connected.*

Proof. Suppose that for some $X \in \mathbb{R}^n$, $\omega(X)$ is compact. Then, by Lemma 2.13, there exists an $s \in \mathbb{R}$ such that $\{\varphi_t(X) \mid t \geq s\}$ is bounded. For all $n \in \mathbb{N}$, let $a_n = s + n$. The sequence (a_n) is then strictly increasing and divergent. Because $a_n > s$ for all n, the sequence $(\varphi_{a_n}(X))$ is bounded. By the Bolzano-Weierstrass Theorem, there exists an increasing divergent sequence (b_n) which is a subsequence of (a_n) such that $(\varphi_{b_n}(X))$ converges to some point $Q \in \mathbb{R}^n$. By Definition 2.7, $Q \in \omega(X)$. Thus, $\omega(X)$ is nonempty.

Suppose, for the sake of contradiction, that $\omega(X)$ is disconnected. Then, there exist disjoint open sets U and V such that $U \cap \omega(X)$ is nonempty, $V \cap \omega(X)$ is nonempty, and

$$(U \cap \omega(X)) \cup (V \cap \omega(X)) = \omega(X).$$

Let $Y_u \in \omega(X) \cap U$ and let $Y_v \in \omega(X) \cap V$. By Definition 2.7, there exist increasing divergent sequences (s_n) and (r_n) such that $\lim_{n\to\infty} \varphi_{s_n}(X) = Y_u$ and $\lim_{n\to\infty} \varphi_{r_n}(X) = Y_v$. Suppose that $N \in \mathbb{N}$ is sufficiently large so that for all $n \geq N$, $\varphi_{s_n}(X) \in U$ and $\varphi_{r_n}(X) \in V$. We will now inductively define an increasing sequence (t_k). Let $t_1 = r_N$ and let $t_2 = s_i$ where $i \geq N$ and $s_i > r_N$. Such an s_i exists because (s_n) is unbounded. In general, if for an even $i \in \mathbb{N}$, t_i has been defined as an element of (s_n), define t_{i+1} to be an element $r_j \in (r_n)$ such that $j \geq N$ and $r_j > t_i$. Similarly, if for an odd $i \in \mathbb{N}$, t_i has been defined as an element of (r_n), define t_{i+1} to be an element $s_j \in (s_n)$ such that $s_j > t_i$ and $j \geq N$. The sequence $(\varphi_{t_k}(X))_{k=1}^{\infty}$ then alternates between being in U and V. Since U and V are disjoint and $\varphi_t(X)$ is continuous, we can choose a sequence (τ_i) with $\varphi_{\tau_i}(X) \in \mathbb{R}^n \setminus (U \cup V)$ and $t_i < \tau_i < t_{i+1}$ for all $i \in \mathbb{N}$. For instance, by Lemma 2.14, we could choose τ_i to be in ∂U or ∂V. Therefore, for all $i \in \mathbb{N}$, there exists a $\tau_i \in (t_i, t_{i+1})$ such that $\varphi_{\tau_i} \notin U$ and $\varphi_{\tau_i}(X) \notin V$ (see Figure 2).

Because, by Lemma 2.13, $\varphi_t(X)$ is bounded for sufficiently large t, the sequence $(\varphi_{\tau_n}(X))$ is bounded. In particular, by the Bolzano-Weierstrass Theorem, there exists a subsequence, (q_n), of (τ_n) such that $(\varphi_{q_n}(X))$ converges to some point $P \in \mathbb{R}^n$. Since $(\varphi_{q_n}(X))$ is contained in the closed set $\mathbb{R}^n \setminus (U \cup V)$, P is not in U or V. By Definition 2.7, $P \in \omega(X)$. This contradicts the supposition that $\omega(X) \subset (U \cup V)$. We have therefore proven that $\omega(X)$ is connected by contradiction. □

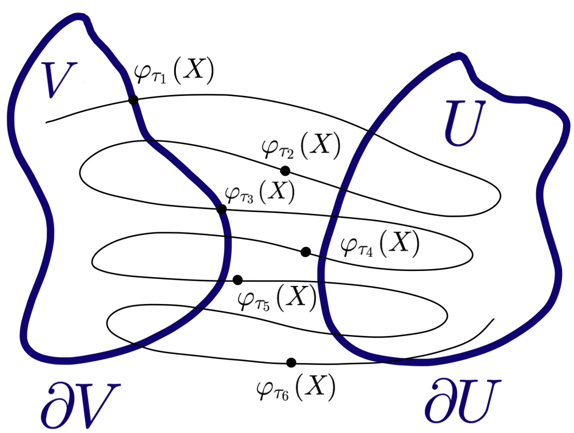

Figure 2

3. Why two dimensions?

Now that we have established some basic facts about smooth dynamics in $\mathbb{R}^n$, it is time to turn our attention to the plane. The Poincaré-Bendixson Theorem restricts how complicated ω and α limit sets in the plane can be. The following lemmas, which are specific to the plane, show why limit sets in $\mathbb{R}^2$ must be relatively simple. In particular, since The Jordan Curve Theorem holds in $\mathbb{R}^2$, if a trajectory ever returns near itself, it must spiral inward since it cannot cross itself. This spiraling behavior severely restricts the possible limiting behavior of the trajectory.

Definition 3.1. Suppose that $X' = F(X)$ is a first order autonomous system in $\mathbb{R}^2$. Suppose that $X_0 \in \mathbb{R}^2$ is not an equilibrium point of the system. Let V be a unit vector based at X_0 which is perpendicular to the vector $F(X_0)$ based at X_0. Define $g : \mathbb{R} \to \mathbb{R}^2$ by $g(z) = X_0 + zV$. Then, $g(\mathbb{R})$ is a line in $\mathbb{R}^2$ which contains X_0 and is perpendicular to $F(X_0)$ based at X_0. Such a line is called the *transverse line* at X_0 (see Figure 3).

Because φ is continuous with respect to initial conditions and $F(X_0) \neq 0$, there exists an open neighborhood around X_0 in $\mathcal{L}(X_0)$ where F is not tangent to $\mathcal{L}(X_0)$. Explicitly, there exists a sufficiently small $\varepsilon > 0$ such that for all $X \in \mathcal{L}(X_0)$ with $\|X - X_0\| < \varepsilon$, $F(X)$ is not tangent to $\mathcal{L}(X_0)$. Furthermore, $F(X)$ points in the same direction away from $\mathcal{L}(X_0)$ as $F(X_0)$ for all such X. Such a line segment centered at X_0 is called a *local section* at X_0.

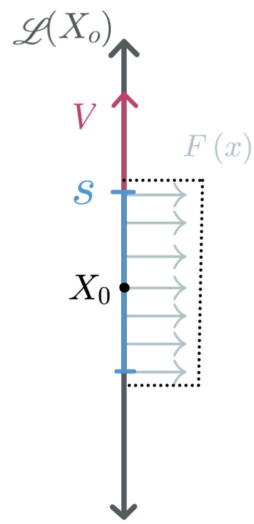

FIGURE 3. A diagram of the transverse line $\mathcal{L}(X_0)$ and a local section S based at X_0. The vector V is the unit vector based at X_0 which is parallel to $\mathcal{L}(X_0)$ and is used to parameterize the transverse line. All of the vectors in the dotted box which are based at points in S all point in the same direction away from $\mathcal{L}(X_0)$ because S is a local section.

Definition 3.2. A finite or infinite sequence $A_0, A_1, A_2, \ldots$ is *monotone along the solution* $\varphi_t(A_0)$ if there exists a non-negative increasing sequence $t_0, t_1, t_2, \ldots$ such that $\varphi_{t_n}(A_0) = A_n$ for all n. We say that a sequence $A_0, A_1, \ldots$ in a local section S is *monotone along* S, if A_i is between A_{i-1} and A_{i+1} on S for all i.

Remark 3.3. A *simple closed curve* in $\mathbb{R}^2$ is a curve which does not intersect itself and which encloses an area. Such curves are sometimes referred to as *Jordan curves*. Simple closed curves separate the plane into two connected components: a bounded interior and an unbounded exterior. Although this fact may seem trivial, proving it requires topological machinery far outside the scope of this paper. A proper treatment of Jordan curves and the Jordan Curve Theorem can be found in Chapter 10 of [3]. We will denote the interior and exterior of a simple closed curve γ as $\text{int}(\gamma)$ and $\text{ext}(\gamma)$ respectively (see Figure 4). It is worth noting that all periodic orbits are simple closed curves.

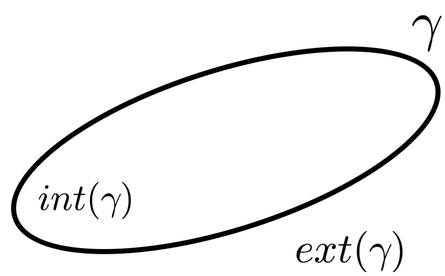

Figure 4

Lemma 3.4. *Suppose that $X' = F(X)$ is a C^1 autonomous system in $\mathbb{R}^2$. Let S be a local section and let $Y_0, Y_1, Y_2, \ldots$ be a sequence of points which lie on S and which are all on the same solution curve $X(t)$. If this sequence of points is monotone along $X(t)$, then it is monotone along S.*

Proof. Let $Y_0, Y_1, \ldots$ be a sequence of points which lie on a section S and which are monotone along a solution curve $X(t)$. Suppose, for the sake of contradiction, that there exists an i such that Y_{i+1} is between Y_i and Y_{i-1}. Let C denote the segment of the curve $X(t)$ which starts at Y_{i-1} and ends at Y_i and let D denote the line segment in S which connects Y_{i-1} and Y_i. The union of C and D is then a simple closed curve (see Figure 5). Furthermore, for all $X \in D$, $F(X)$ points away from the interior of $C \cup D$. Since $\varphi_t(Y_i)$ cannot intersect C and cannot enter the interior of $C \cup D$ through D, $\varphi_t(Y_i)$ remains in the exterior of $C \cup D$ for all $t \geq 0$. Because $Y_0, Y_1, \ldots$ is monotone along $X(t)$, there exists a positive $s \in \mathbb{R}$ such that $\varphi_s(Y_i) = Y_{i+1}$. This implies that $Y_{i+1} \in \text{ext}(C \cup D)$ which contradicts the fact that $Y_{i+1} \in D$. ☐

Lemma 3.5. *If $Y \in \omega(X)$ or $Y \in \alpha(X)$ for some $X \in \mathbb{R}^2$, then $\varphi_t(Y)$ crosses any local section no more than once.*

Proof. Suppose that $Y \in \omega(X)$ and that $\varphi_t(Y)$ cross a local section S at two distinct points Y_1 and Y_2. Let O_1 and O_2 be open neighborhoods of Y_1 and Y_2 which are disjoint. By Definition 2.7, there exist increasing divergent sequences (t_n) and (s_n) such that $\lim_{n\to\infty} \varphi_{t_n}(X) = Y_1$ and $\lim_{n\to\infty} \varphi_{s_n}(X) = Y_2$. Thus, there exist infinitely many arbitrarily large t_n and s_n such that $\varphi_{t_n}(X) \in O_1$ and $\varphi_{s_n}(X) \in O_2$. We can therefore find a finite sequence

$$t_{n_1}, s_{n_2}, t_{n_3}$$

which is strictly increasing such that $\varphi_{t_{n_1}}(X), \varphi_{t_{n_3}}(X) \in O_1$ and $\varphi_{s_{n_2}}(X) \in O_2$ (see Figure 6). The sequence

$$\varphi_{t_{n_1}}(X), \varphi_{s_{n_2}}(X), \varphi_{t_{n_3}}(X)$$

is therefore monotone along $X(t)$ but is not monotone along S. This contradicts Lemma 3.4. This argument can be straightforwardly adapted in the case that $Y \in \alpha(X)$. We have thus shown that if Y is in a limit set, then the solution through Y crosses any local section at no more than one point. ☐

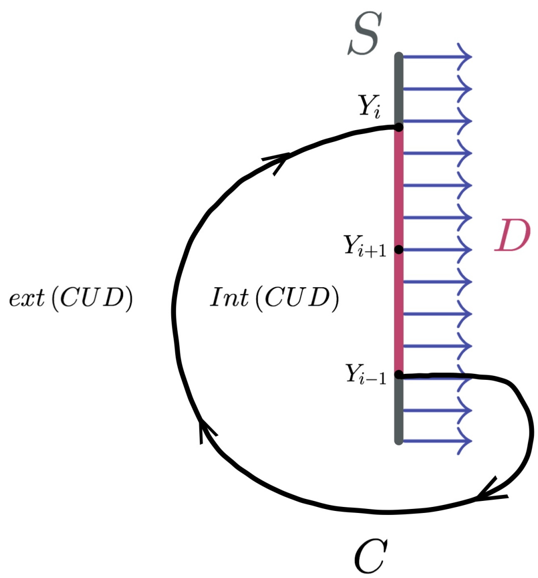

Figure 5

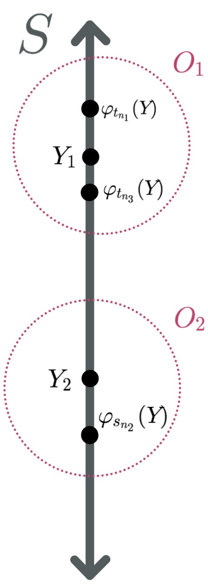

Figure 6

4. The Poincaré-Bendixson Theorem

Theorem 4.1 (Poincaré-Bendixson Theorem)**.** *Let $X' = F(X)$ be a C^1 autonomous system in $\mathbb{R}^2$. Suppose that $X \in \mathbb{R}^2$ such that $\omega(X)$ is compact. If $\omega(X)$ contains no equilibrium points, then $\omega(X)$ is a closed orbit. Similarly, if $\alpha(X)$ is compact and contains no equilibrium points, then $\alpha(X)$ is a closed orbit.*

Proof. Suppose that $\omega(X)$ is compact and contains no fixed points. By Lemma 2.13, $\omega(X)$ is nonempty. Let $P \in \omega(X)$. Since ω-limit sets are closed and invariant, by Lemma 2.12, $\omega(P) \subset \omega(X)$. Furthermore, since $\omega(P)$ is a closed subset of a compact set, it is compact and thus nonempty by Lemma 2.13.

Let $Q \in \omega(P)$. Because $\omega(P) \subset \omega(X)$ and $\omega(X)$ contains no equilibrium points, Q is not an equilibrium point. We can then let S be a local section at Q. Let (t_n) be an increasing divergent sequence such that

$\lim_{n\to\infty} t_n = \infty$ and $\lim_{n\to\infty} \varphi_{t_n}(P) = Q$. Let

$$\mathcal{G} = \{\varphi_t(V) \mid V \in S \text{ and } t \in [-a, a]\}$$

where $a > 0$ is sufficiently small so that every trajectory which enters the set $\mathcal{G}$ eventually crosses S and then exits the set $\mathcal{G}$ (see Figure 7). Such an a exists because F is continuous. Further justification for the existence of the set $\mathcal{G}$ can be found in Chapter 10.2 of [2].

The set $\mathcal{G}$ is a connected set whose interior is an open neighborhood of Q. Since $(\varphi_{t_n}(P))$ gets arbitrarily close to Q, there exists some $k \in \mathbb{N}$ such that for all $m \geq k$, $\varphi_{t_m}(P) \in \mathcal{G}$. By the definition of $\mathcal{G}$, there exists a $V \in S$ and an $r \in (-a, a)$ such that $\varphi_r(V) = \varphi_{t_k}(P)$. By Definition 2.1, $\varphi_{t_k - r}(P) = V$. Moreover, the trajectory $\varphi_t(P)$ leaves $\mathcal{G}$ at time $t_k - r + a$. However, there must exist a time $t_i \in (t_n)$ such that $t_i > t_k - r + a$ and $\varphi_{t_i}(P) \in \mathcal{G}$. Furthermore, by the definition of $\mathcal{G}$, there exists an $\alpha \in [t_i - a, t_i + a]$ such that $\varphi_\alpha(P) \in S$. Since $P \in \omega(X)$, by Lemma 3.5, $\varphi_t(P)$ only ever crosses S at V. Therefore, $\varphi_\alpha(P) = V$. Because

$$\alpha \geq t_i - a > t_k - r$$

and $\varphi_\alpha(P) = \varphi_{t_k - r}(P)$, by Definition 2.5, P lies on a closed orbit.

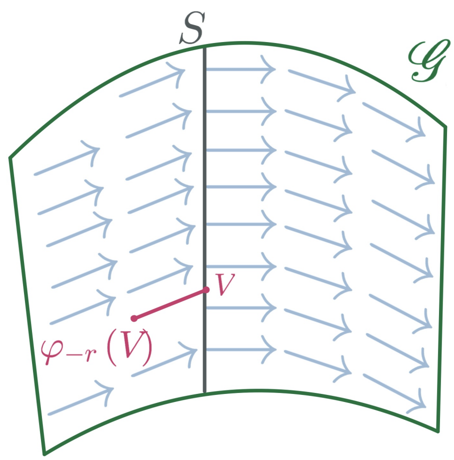

Figure 7

We have thus shown that every $P \in \omega(X)$ lies on a closed orbit Γ_P. Furthermore, by Remark 2.8 and Lemma 2.12, for all $P \in \omega(X)$, $\Gamma_P \subset \omega(X)$.

Let $Y \in \omega(X)$. Suppose, for the sake of contradiction, that for all $\varepsilon > 0$,

$$\{X \in \mathbb{R}^2 \mid d(X, \Gamma_Y) < \varepsilon\} \cap (\omega(X) \setminus \Gamma_Y) \neq \emptyset,$$

where $d(X, \Gamma_Y)$ denotes the distance between the point X and the set Γ_Y. Since Γ_Y is compact, F is uniformly continuous on Γ_Y. In particular, there exists a sufficiently small $\delta > 0$ such that for all $Z \in \Gamma_Y$, there is a section centered at Z with length 2δ. Let T be an element of the set

$$\left\{X \in \mathbb{R}^2 \mid d(X, \Gamma_Y) < \frac{\delta}{2}\right\} \cap (\omega(X) \setminus \Gamma_Y).$$

Then there exists a section S_Z centered at a point $Z \in \Gamma_Y$ which contains T. Since $T \in \omega(X)$, T lies on a periodic orbit Γ_T which is a subset of $\omega(X)$ (see Figure 8). Let (a_k) be an increasing divergent sequence such that $(\varphi_{a_k}(X))$ approaches Z. Likewise, let (b_k) be an increasing divergent sequence such that $(\varphi_{b_k}(X))$ converges to T. Because $(\varphi_{a_k}(X))$ converges to Z, there exist real numbers c_1 and c_2 such that $c_1 < c_2$ and $\varphi_{c_1}(X), \varphi_{c_2}(X) \in S$ with $\varphi_{c_2}(X)$ closer to Z. Furthermore, since (b_k) is unbounded, there exists a $c_3 \in \mathbb{R}$ such that $c_1 < c_2 < c_3$ and $\varphi_{c_3}(X) \in S$ with

$$\|\varphi_{c_3}(X) - T\| < \|\varphi_{c_1}(X) - T\|.$$

All three points, $\varphi_{c_1}(X), \varphi_{c_2}(X)$, and $\varphi_{c_3}(X)$ must be between T and Z since $\varphi_t(X)$ cannot cross Γ_T or Γ_Y (see Figure 9). The finite sequence $\varphi_{c_1}(X), \varphi_{c_2}(X), \varphi_{c_3}(X)$ is then monotone along the solution $\varphi_t(X)$ and not monotone along the section S. This contradicts Lemma 3.4. Therefore, there exists an $\varepsilon > 0$ such that

$$\{X \in \mathbb{R}^2 \mid d(X, \Gamma_Y) < \varepsilon\} \cap (\omega(X) \setminus \Gamma_Y) = \emptyset.$$

Suppose that $\omega(X) \neq \Gamma_Y$. Let $A = \{X \in \mathbb{R}^2 \mid d(X, \Gamma_Y) < \frac{\varepsilon}{2}\}$ and let B denote the interior of the complement of A. In particular, A and B are disjoint open sets such that $\omega(X) \subset A \cup B$. Therefore, $\omega(X)$ is disconnected. However, this contradicts Lemma 2.15. Thus, $\omega(X) = \Gamma_Y$. We have therefore proven that if $\omega(X)$ is compact and contains no equilibrium points, then $\omega(X)$ is a single periodic orbit.

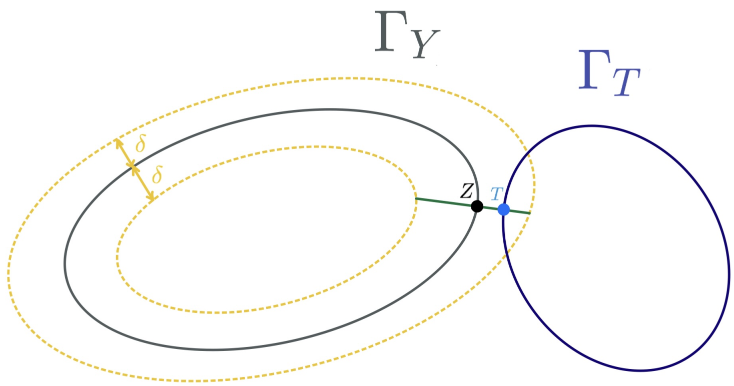

FIGURE 8

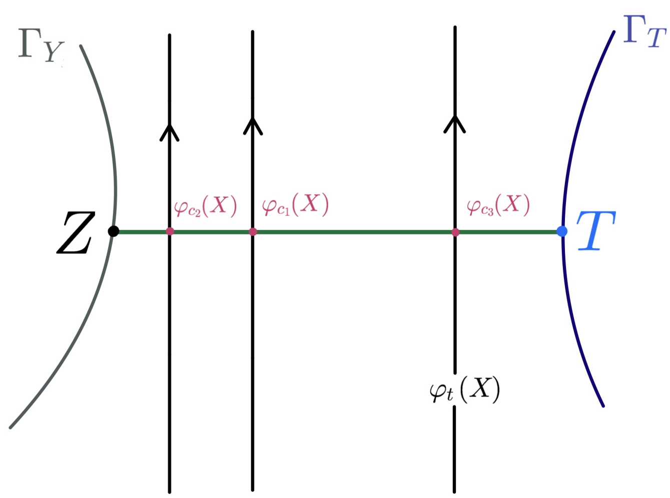

FIGURE 9

□

Definition 4.2. A periodic orbit γ is a *limit cycle* if there exists an X which is not in γ such that $\gamma \subset \omega(X)$ or $\gamma \subset \alpha(X)$.

Corollary 4.3. *Suppose that γ is a closed orbit such that $X \notin \gamma$ and $\omega(X) = \gamma$. Then, there exists an open neighborhood U of X such that for all $Y \in U$, $\omega(Y) = \gamma$. Furthermore, the set $\{X \mid \omega(X) = \gamma\} \setminus \gamma$ is open.*

Proof. A proof and thorough explanation of this corollary can be found in Chapter 10 of [2]. □

Corollary 4.4. *If U is the interior of a closed orbit γ, then U contains an equilibrium point.*

Proof. The following proof is a modified version of the proof given in Chapter 10 of [2]. Suppose, for the sake of contradiction, that U is an open set which is the interior of a periodic orbit γ such that U contains no equilibrium points or limit cycles. The set $U \cup \gamma$ is compact and invariant and therefore contains the ω and α limit sets of every one of its elements. By The Poincaré-Bendixson Theorem, the ω and α limit sets of the points of $U \cup \gamma$ must be γ since there are no other equilibrium points or closed orbits. Let $X \in U \cup \gamma$ and let $Y \in \gamma$. Let S be a section at Y. Then, there exist sequences (a_n) and (b_n) with $\lim_{n\to\infty} a_n = \infty$ and $\lim_{n\to\infty} b_n = -\infty$ such that

$$\lim_{n\to\infty} \varphi_{a_n}(X) = \lim_{n\to\infty} \varphi_{b_n}(X) = Y,$$

with $\varphi_{a_n}(X), \varphi_{b_n}(X) \in S$ for all n. However, this contradicts the monotonicity result of Lemma 3.4. We have thus shown that U must contain either an equilibrium point or a limit cycle.

Now suppose that U contains no equilibrium points. If U contains a finite number of closed orbits, then there exists a closed orbit $\Gamma \subset U$ which encloses the least area. The interior of Γ must then contain an equilibrium point, however this contradicts our assumption that U does not contain any equilibrium points.

Now suppose that U contains infinitely many limit cycles. Let a be the infimum of the set of areas enclosed by periodic orbits in U. Let (Γ_N) be a sequence of periodic orbits in $\gamma \cup U$ such that the sequence (a_n) of areas enclosed by each Γ_n satisfies $\lim_{n\to\infty} a_n = a$. Let (Q_n) be a sequence of points such that for all n, $Q_n \in \Gamma_n$. Since (Q_n) is contained in a compact set, there is a subsequence (T_n) which converges to a point $T \in U \cup \gamma$. Suppose that T does not lie on a closed orbit. By The Poincaré-Bendixson Theorem, the solution through T must approach a limit cycle. By Corollary 4.3, there exists some T_m which also approaches that same limit cycle. This is a contradiction, as T_m already lies on a closed orbit and therefore cannot approach a limit cycle. Therefore, T must lie on a closed orbit.

This implies that the sequence (Γ_n) approaches a closed orbit Γ_T which encloses the minimum area a. Thus, a is non-zero and there cannot be any more closed orbits in the interior of Γ_T. This is a contradiction, as we have already showed that the interior of closed orbits must contain either a periodic orbit or an equilibrium point. We have therefore proven that the interior of γ must contain an equilibrium point by contradiction. □

5. Application in Neuroscience

We will now use the Poincaré-Bendixson Theorem to analyze a system of differential equations used to model the excitability of a single neuron. The two dimensional system that we will discuss is due to Fitzugh and Nagumo and it is a reduction of the much more complex four dimensional Hodgkin-Huxley model [4]. The Fitzhugh-Nagumo model is given by the following pair of ordinary first order differential equations:

$$\frac{dx}{dt} = x - \frac{1}{3}x^3 - y + I \tag{5.1}$$

$$\frac{dy}{dt} = \epsilon(bx - y + a) \tag{5.2}$$

where $\epsilon << 1$, $0 < a < 1$, and $b > 1$. We will denote the vector field $\begin{bmatrix} x' \\ y' \end{bmatrix}$ as $V(x, y)$. The function $x(t)$ represents the membrane potential of the neuron and the function $y(t)$ represents the excitability of the neuron. The parameters a, b, and ϵ represent fixed physical and biological properties of the cell and I is the current being put into the neuron. A more detailed mathematical analysis of this model can be found in [4].

In order to simplify our analysis we will only examine a single interesting case and fix our parameters in the following way:

$$\begin{aligned} \epsilon &= 0.08 \\ a &= 0.3 \\ b &= 6 \\ I &= 1. \end{aligned}$$

In this case there is a single equilibrium point at approximately $(0.14, 1.14)$ which is unstable and a source. Justification for why this equilibrium is an unstable source can be found in Chapter II of [4].

We will now use the Poincaré-Bendixson Theorem to find a limit cycle of the system. In order to prove the existence of a limit cycle, we will construct a closed positively invariant region in the plane. Because the equilibrium point is a source, we can find a small open neighborhood O of the equilibrium point such that on ∂O, the vector field V points away from O. Let L denote the line defined by the equation $y' = 0$. Let A be the point where the vertical line $x = 20$ intersects L and let B denote the point where the vertical line $x = -20$ intersects L. Let K denote the filled in closed rectangle whose diagonal is the segment connecting A to B. We can also safely assume that O is sufficiently small to be contained within the interior of K.

We will now show that $K \setminus O$ is a positively invariant set. The line L divides the plane into two regions in which $y' > 0$ in one and $y' < 0$ in the other (see Figure 11). Because the upper boundary of K is in the region where $y' < 0$ and the lower boundary is in the region where $y' > 0$, V points towards the interior of K on the upper and lower line segments bounding it. For all points (x, y) on the boundary of K, $y \in (-121, 121)$. Therefore, $x' < 0$ on the right hand edge of K and $x' > 0$ on the left hand boundary of K. Thus on the boundary of K and on the boundary of O, V points towards the interior of $K \setminus O$. In particular, any trajectory that starts within $K \setminus O$ can not leave the set. Therefore, by Definition 2.6, $K \setminus O$ is positively invariant.

Let $X \in K \setminus O$. Since $K \setminus O$ is closed and positively invariant, by Lemma 2.12, $\omega(X) \subset K \setminus O$. This means that $\omega(X)$ is bounded and does not contain an equilibrium point. In particular, by the Poincaré-Bendixson Theorem, $\omega(X)$ must be a closed orbit. We have thus proven the existence of a closed orbit, which would have been difficult to find as an explicit solution to (5.1) and (5.2). Furthermore, this closed orbit is a limit cycle that every point in $K \setminus O$ spirals towards (see Figure 10).

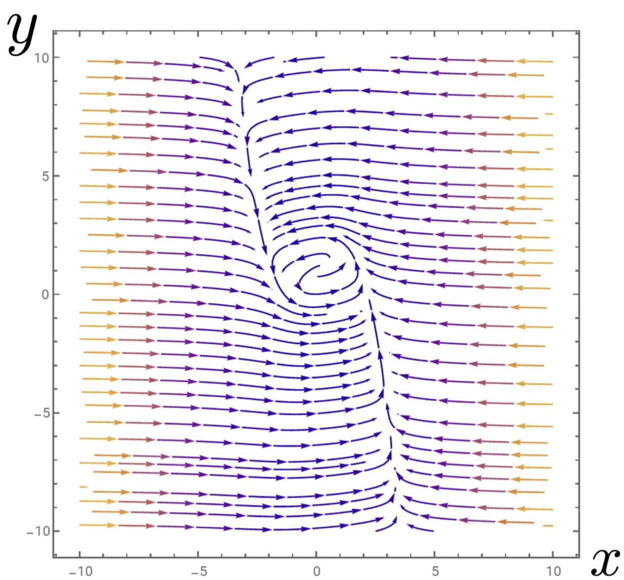

FIGURE 10. Vector field plot of the Fitzhugh-Nagumo Model with $\epsilon = 0.08$, $a = 0.3$, $b = 6$, and $I = 1$.

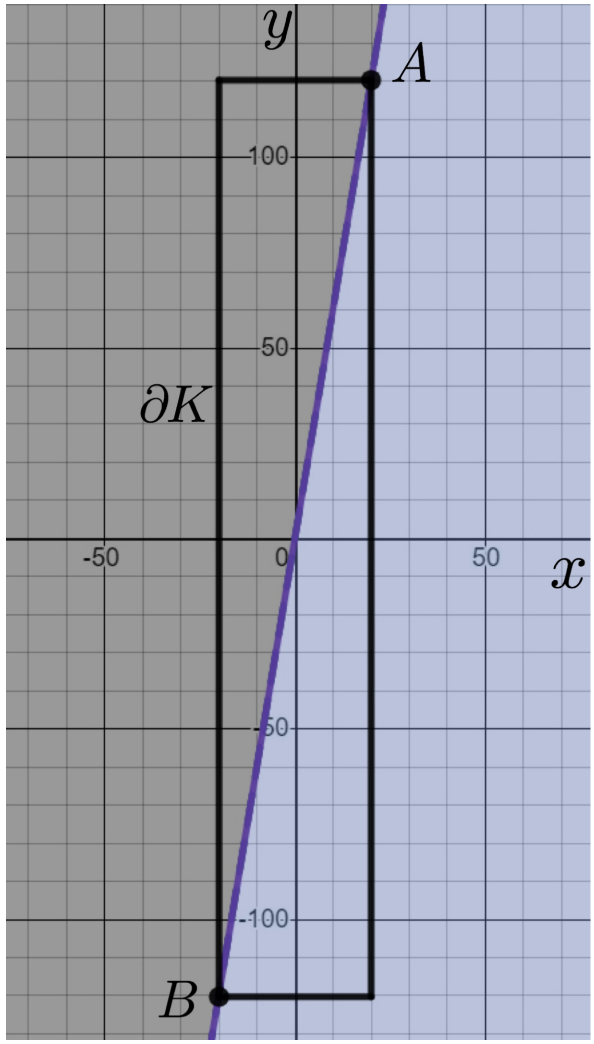

FIGURE 11. A plot of the region K. The purple line is the line given by the equation $y' = 0$. The blue shaded region is where $y' > 0$ and the grey shaded region is where $y' < 0$.

6. The Brouwer Fixed Point Theorem

We will now use the Poincaré-Bendixson Theorem to prove the generalized Brouwer Fixed Point Theorem in two dimensions. The following proof is a modified and generalized version of the proof given in [5].

Theorem 6.1 (Brouwer Fixed Point Theorem)**.** *Suppose that $A \subset \mathbb{R}^2$ is a convex and compact set and that $F : A \to A$ is continuous. Then, there exists an $X \in A$ such that $F(X) = X$.*

Proof. Let $A \subset \mathbb{R}^2$ be convex and compact and let $F : A \to A$ be continuous, where

$$F(x, y) = \begin{bmatrix} f(x, y) \\ g(x, y) \end{bmatrix}.$$

Let $\varepsilon > 0$. By the Weierstrass-Approximation Theorem, there exist polynomials $p(x, y)$ and $q(x, y)$ such that

$$\left\| \begin{bmatrix} f(x, y) \\ g(x, y) \end{bmatrix} - \begin{bmatrix} p(x, y) \\ q(x, y) \end{bmatrix} \right\| < \varepsilon$$

for all $(x, y) \in A$. Suppose that F does not have any fixed points on the boundary of A. Let $Y \in \partial A$. The vector which points from Y to $F(Y)$ has a non-zero length and, since A is convex, points into A (see Figure

12). We can also assume that our polynomial approximation is close enough so that for all $Y \in \partial A$, the vector from Y to $\begin{bmatrix} p(Y) \\ q(Y) \end{bmatrix}$ is never 0 and points into the set A. In particular, the vector field

$$G(x, y) = \begin{bmatrix} p(x, y) \\ q(x, y) \end{bmatrix} - \begin{bmatrix} x \\ y \end{bmatrix}$$

points into A at every point on the boundary of A. Therefore, A is a positively invariant set by Definition 2.6. By The Poincaré-Bendixson Theorem, A must either contain an equilibrium point or a closed orbit γ. Note that we can apply the Poincaré-Bendixson Theorem because G is C^1. In the case that A has a periodic orbit γ, the interior of γ must have an equilibrium point by Corollary 4.4. Therefore, A contains an equilibrium point of G in all cases. Let (a, b) denote this equilibrium point. Thus, $p(a, b) = a$ and $q(a, b) = b$. In particular,

$$\left\| \begin{bmatrix} f(a, b) \\ g(a, b) \end{bmatrix} - \begin{bmatrix} a \\ b \end{bmatrix} \right\| = \left\| \begin{bmatrix} f(a, b) \\ g(a, b) \end{bmatrix} - \begin{bmatrix} p(a, b) \\ q(a, b) \end{bmatrix} \right\| < \varepsilon.$$

We have therefore proven that for all $\varepsilon > 0$, there exists a point $P \in A$ such that $\|F(P) - P\| < \varepsilon$.

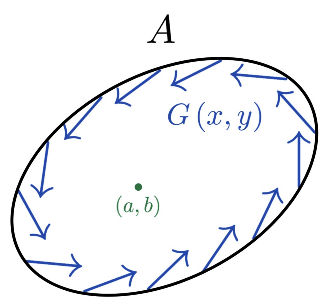

FIGURE 12

Let (Y_n) be a sequence of points in A such that for all $n \in \mathbb{N}$, $\|F(Y_n) - Y_n\| < \frac{1}{n}$. Since (Y_n) is contained in A, which is compact, there exists a subsequence, (Z_n), of (Y_n) which converges to a point $Z \in A$. By the continuity of F,

$$\lim_{n\to\infty} F(Z_n) = F(Z).$$

Thus for every $\varepsilon > 0$, there exists a sufficiently large $N \in \mathbb{N}$ such that

$$\begin{aligned} \|F(Z) - Z\| &\leq \|F(Z) - Z_N\| + \|Z_N - Z\| \\ &\leq \|F(Z) - F(Z_N)\| + \|F(Z_N) - Z_N\| + \|Z_N - Z\| \\ &< \varepsilon. \end{aligned}$$

Therefore $F(Z) = Z$ and Z is a fixed point of F. □

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